ELECTROMAGNETIC RESPONSE OF TIME-REVERSAL BREAKING METALLIC PHASES IN TWO DIMENSIONS

BY

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DISSERTATION

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Abstract

We explore two phases in two-dimensional electron fluids in which the time-reversal symmetry is broken spontaneously by using the method of higher dimensional bosonization. Mean-field calculations show that the order parameter is two two-component real vectors [Sun and Fradkin, 2008]. There are two phases: the β phase in which the two order parameters are perpendicular and the α phase in which they are parallel. This β phase exhibits nonvanishing un-quantized anomalous Hall effect in the absence of external magnetic fields, which corresponds to the Berry curvature on the Fermi surface. The α phase does not have that property. To go beyond the mean-field, we introduce the machinery of higher dimensional bosonization. Our preliminary results show that in the mean field limit of the bosonized theory, the fluid spontaneously transforms into the time-reversal broken phase. It is evident from the result that the critical point we have is similar to that of Pomeranchuk instability. The quartic term in the dispersion expansion needed to be introduced to stabilize the theory. The correction coming from the higher order terms introduces the coupling to the curvature of the Fermi surfaces. The β_1 phase in the bosonized picture is not correct so we go back to the fermionic theory and integrate out the fermions in the symmetric phase directly to achieve an effective action, $\mathcal{S}_{\text{eff}}[\Gamma_{\mu i}, \mathbf{A}]$. Finally, the full response polarization tensor is derived and its Ward identity is shown to be obeyed.

To my father, to my mother and to my supportive wife, ***Mahalia***.

Acknowledgments

First and foremost, I want to express my deep appreciation for my committee members. Professor Nadya Mason has guided my thought process throughout my doctoral work and has always asked insightful questions that have pushed me to deepen my understanding. I have been inspired by her eye for applications. Professor Thomas Faulkner expanded my intellectual horizons by conversations about his cross-community work in Physics and his clear excitement for future opportunities. In addition, I am grateful to Professor Shinsei Ryu for generously being the chair of my committee. I will be forever touched by the love of physics he exudes. Finally, I wish to express my deepest gratitude to my advisor, Professor Eduardo Fradkin, for his support through both his brilliant insights and his time. I also would like to thank him for patiently listening and answering a lot of my naïve questions over the years. Under his generous tutelage, I have grown both as a researcher and as a human being. He has been my biggest advocate, for which I will be always grateful. I hope that I may be able to give back to him and his academic family for the rest of my life.

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List of Abbreviations

AHE	Anomalous Hall effect.
C	Chiral symmetry.
DC	Direct current.
DOS	Density of state.
ELC	Electronic liquid crystal.
OPE	Operator product expansion.
P	Pairity symmetry.
T	Time-reversal symmetry.

Chapter 1

Introduction

Understanding electronics properties of metal is crucial to the development of new knowledge in modern condensed matter physics. It often also plays an important role in the advancement of technology. This thesis focuses on the study of unquantized Anomalous Hall conductance in the absence of external magnetic fields resulting from a spontaneous time-reversal symmetry breaking. This phenomena arises in two-dimensional electron fluids undergoing a phase transformation through a Pomeranchuk-type critical point. The phase under investigation (the β_1 phase) is not a conventional nematic phase because the Fermi surfaces are isotropic but the phase difference between the fermions has non-trivial winding, hence breaks rotational invariant. A competing phase, α_1, is an expected nematic phase for $l = 2$. A system could go into either phase depending on the parameters of the free energy. Furthermore, it has a non-trivial phase winding about the Fermi surfaces. To understand the electromagnetic response of this system, we will develop some theoretical tools based on standard techniques such as bosonization and diagrammatic field theory.

1.1 Anomalous Hall Effect

It is important to make a clear distinction between the model in question and the conventional anomalous Hall effect.

The Hall effect has been a crucial tool in advancing our understanding in the physics of electronics particularly in the field of electronic transport. Edwin Hall (Hall, 1879) made an important discovery that when one passes current through a conductor under a perpendicular magnetic field, Lorentz force bends the trajectory of the charges so that they accumulate on one side of the conductor resulting in a voltage transverse to the direction of the current and the external magnetic field. This discovery provided a tool to more accurately measure the carrier concentration in non-magnetic conductors ([Nagaosa et al., 2010])

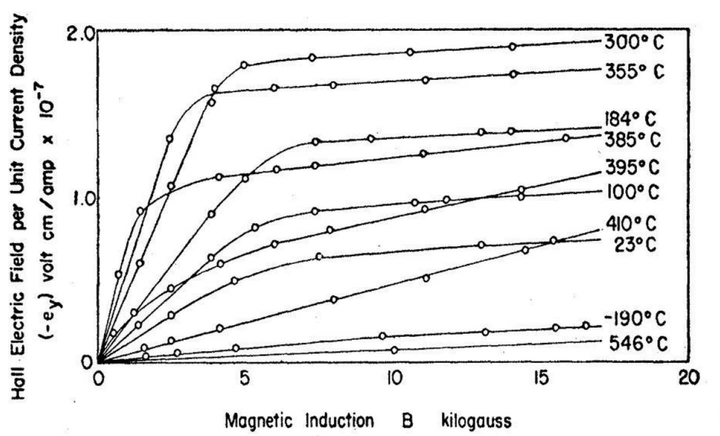


Figure 1.1: The dependence of the Hall resistivity, $\rho_{x,y}$ on the applied magnetic field, H_z in Ni. The dependence saturates at some high value and looks independent of H_z [Nagaosa et al., 2010].

Historically, experimentalists knew very early on that the Hall resistivity measured in non-magnetic and in ferromagnetic metals depend differently on the applied magnetic field. In non-magnetic conductors, the Hall resistivity, $\rho_{x,y}$, increases linearly with the applied field, H_z, as expected from the Lorentz force law. However, in the case of ferromagnetic conductors, $\rho_{x,y}$ increases steeply in the weak-field region but later plateaus off at some high-value that is nearly independent of H_z (see Figure 1.1).

From the experiment by Pugh and collaborators ([Pugh, 1930]; [Pugh and Lippert, 1932]), the resistivity is found to have the following independence:

$$\rho_{x,y} = R_0 H_z + R_s M_z \tag{1.1}$$

where M_z is the spontaneous magnetization. From this, we can conclude that in ferromagnetic conductors there is a contribution of the Hall conductivity coming from an intrinsic magnetic property of the material due to spin-orbit coupling. The contribution to Hall Conductance that is independent of the external magnetic field is called anomalous Hall conductance (σ_{xy}^{AH}). There are three contributions to σ_{xy}^{AH}: the skew-scattering, side-jump, and intrinsic. We will limit ourselves to the intrinsic contribution here. The intrinsic contribution, $\sigma_{xy}^{AH-intr}$, is topological in nature. Specifically, it can be related to the integration over the Fermi sea of the Berry's curvature of each occupied band, or equivalently, to the integral of Berry phases over cuts of Fermi surface segments ([Haldane, 2004]; [Wang et al., 2007]). In this regime, one can think of the **AHE** as the unquantized equivalence of the quantum Hall effect. In the β_1 phase, the time-reversal symmetry is broken spontaneously through an interaction instead of due to an intrinsic magnetic order.

1.2 Nematic Fermi Fluid

As previously discussed, the β_1 phase is driven by an instability through a Pomeranchuk-type critical point. This is similar to the mechanism for the nematic phase, which is a type of an electronic liquid crystal phase.

Liquid crystal phase is a phase with properties that border those of a liquid and those of a crystalline solid, as the name suggests. The properties include an ability to flow like a liquid, an inability to support shear, and the formation of droplets. Its optical, electrical, and magnetic properties exhibit anisotropy, which are signature of crystalline solids. The optical properties, in particular birefringence, are used to distinguish between different types of liquid crystals, namely: nematic, cholesteric, smetics.

An interested reader can study an extensive review of an equivalence phase in electronics system by Fradkin ([Fradkin, 2010]).

Nematic phase is a type of electronic liquid crystal (ELC) phases. ELC phases are states of correlated electron fluids that spontaneously break either translational invariance or rotational invariance. The classification based on the classical liquid crystals are:

1. *Crystalline phases*: phases that break all continuous translation symmetries and rotational invariance.

2. *Smectic phases*: phases that break one translation symmetry and rotational invariance.

3. *Nematic and hexatic phases*: uniform (liquid) phases that break rotational invariance.

4. *Isotropic*: uniform and isotropic phases.

The phase that is the main interest of the current study is called β_1 phase. This phase is due to the system undergoing a phase transformation through a Pomeranchuk-type critical point. Furthermore, depending on the parameters of the free energy ([Sun and Fradkin, 2008]), the system could be in the α_1 phase, which is essentially a nematic phase for a two-band case, or the β_1 phase, which has isotropic Fermi surfaces but non-trivial relative phase winding. The theory has a similar flavor to the theory formulated to study the instabilities of Fermi fluids in the spin channel ([Wu et al., 2007]). A key theoretical basis of this work is the microscopic theory of nematic Fermi fluids ([Oganesyan et al., 2001]).

1.3 Pomeranchuk Instability

The main mechanism for breaking the time-reversal symmetry spontaneously is through a Pomeranchuk-type instability in some channels of the interaction. Hence in this section the basic of Pomeranchuk instability is reviewed. The system is that of a Fermi Liquid with interaction parameterized by the Landau Fermi liquid parameter. These instabilities were first described by Pomeranchuk in 1958[Pomeranchuk, 1958]. Pomeranchuk instabilities are related to broken symmetries and deformations of the Fermi surface in Fermi liquid systems. Pomeranchuk instabilities are associated with collective modes of such systems and large fluctuations about the quantum ground state. In modern context, these instabilities are considered quantum critical points [Sachdev, 1999] in parameter space. This quantum criticality is a source of many current excitements in condensed matter physics such as non-Fermi-liquid and quantum criticality in high-T_c cuprate(for example [Sebastian et al., 2010], [Castellani et al., 1996], [Slooten et al., 2009], [Jiang et al., 2009]).

Pomeranchuk instability occurs in a particular angular momentum channel in which the interaction becomes sufficiently large and negative enough to destabilize the Fermi surface. The Fermi surface then become "soft" with respect to to a particular deformation, indicating a thermodynamic instability [Oganesyan et al., 2001]. This causes an Isotropic Fermi surface to spontaneously distorts into an anisotropic Fermi surface which breaks rotational symmetry. The exact value of the Landau Fermi liquid parameter differs in different dimensions. A detailed derivation of this critical point for three dimension can be studied in [Baym and Pethick, 1991]. An example of this is in a system of two-dimensional nematic fermi fluids. The effective free energy contains a quadratic term which start to become negative when

$$f_l = -\frac{1}{N(0)} \tag{1.2}$$

where f_l is the Landau Fermi liquid parameter in the angular momentum channel, l. Since the quadratic term is negative, the system becomes unstable and required higher-order terms in the dispersion to stabilize the system. Therefore the system spontaneously transforms to a phase that has an anisotropic Fermi surface. A familiar example of a Pomeranchuk instability is the ferromagnetic (Stoner) instability([Stoner, 1947], [Nagaosa, 1999], [Mattis, 1981],

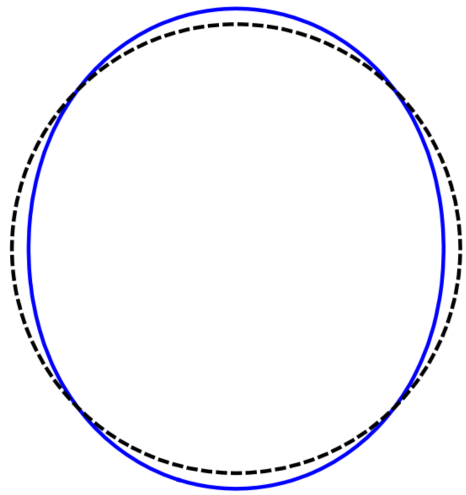


Figure 1.2: An illustration of an Isotropic Fermi surface is spontaneously distorted into a new surface configuration that breaks the rotational symmetry. The instability occurs in the angular momentum channel, $l = 2$. [Fradkin, 2010]

[Yosida, 1996]). The condition is $F_0^a \to -1$; here, the Fermi surface splits into a spin-up surface and a spin-down surface, magnetic susceptibility diverges, and time-reversal symmetry is broken. Also in the $l = 0$ (s-wave) channel, $F_0^s \to -1$ marks an approach to a charge density instability. Another example of this for the case of $l = 2$ can be seen in figure. 1.2. In the work that will be explored in this thesis, the situation is a lot more complicated with two Fermi surfaces and inter-band interaction potential but it will be shown later that the critical point is that of a modified Pomeranchuk critical point. Any readers who are interested to learn more about the Pomeranchuk instability in details can look at the original paper by Pomeranchuk([Pomeranchuk, 1958]). There are many other works that use this mechanism to study non-Fermi liquid properties and other exotic properties (see for example [Maslov and Chubukov, 2010], [Metzner et al., 2003], [Löhneysen et al., 2007]).

1.4 Anomalous Hall effect (AHE) from spontaneous breaking of time-reversal symmetry

This anomalous Hall effect occurs in metals with the broken time-reversal symmetry. It is observed in many materials, such as spinel, $CuCr_2Se_4$, and Heusler alloy, MnSi. Typically, the symmetry is broken explicitly due to external magnetic fields or some intrinsic ferromagnetic order. This is a new and exciting area of research with a lot of interest recently since its connection to the Berry phase links it to the topological nature of currents. Furthermore, the conductance is unquantized, which is unlike the case of the integer and fractional quantum Hall effect usually seen in topological insulators([Stormer, 1992], [Thouless et al., 1982], [Haldane et al., 1987], [Chakraborty, 1988],[Stone, 1992]). An example of an interesting material that might bear some similarities to the system we will be focus on is MnSi. In MnSi, carrier spins are coupled to the chiral spin texture in the helical magnetization. Experimental results and some theoretical work on the material suggest that the above mechanism gives rise to a spontaneous magnetization that breaks time-reversal symmetry [Nagaosa et al., 2010], [Onoda and Nagaosa, 2003].

Intuitively, one could imagine that in some systems that have similar intrinsic properties that could gives rise to a multi-band model, time-reversal symmetry could be broken spontaneously due to condensates in the particle-hole channel that couples the bands. Models of this type then should exhibit the anomalous Hall effect simultaneously. This is still an open area of research with a lot of interest. However, systems with spontaneous breaking of time-reversal symmetry have not yet been observed in experiments. As a first milestone towards these future experimental investigations, I will explore a theoretical framework for models that do break the time reversal symmetry spontaneously through inter-band interactions.

In this thesis, I will put forth a theoretical foundation to understand anomalous Hall effect in two-dimensional metals that does not arise from spontaneous magnetization. The electromagnetic response of models of nematic non Fermi-liquids previously proposed in

Ref.[Sun and Fradkin, 2008] are re-examined using multi-dimensional bosonization and conventional many-body methods. Nematic phases of this model are described by two 2-component real vectors which express the isotropy breaking nematicity in two Fermi surfaces. Of interest is the time-reversal symmetry breaking nematic phase with a non-vanishing unquantized spontaneous anomalous Hall effect at zero external magnetic field. I further present its geometrical description as a Berry phase([Berry, 1985]).

The thesis first covers an overview of the key foundational concepts. In the next chapter, the two-band metal is elaborated. The core results of the mean-field approach is also reviewed ([Sun and Fradkin, 2008]). The third chapter focuses on multi-dimensional bosonization. I show that in the proposed framework the leading order correction from the interaction to the Pomeranchuk condition comes naturally with little effort. In addition, the ground state for the β_1 phase and its response is obtained in the mean-field limit. The chapter also includes a discussion why this approach is incomplete. In the fourth chapter, I use a diagrammatic field theoretic approach to derive an effective field theory and then the full response function. I conclude with an overview of the key contributions and ideas for future work.

Chapter 2

Model

In this chapter the model that will be the starting point of this study is carefully stated. The order parameters that signifies the strength of the time-reversal invariance breaking is also defined. Furthermore an analysis of the underlying symmetries is presented. Lastly the results from mean-field theory analysis(see Ref. [Sun and Fradkin, 2008]) is shown as a motivation for later chapters.

2.1 Microscopic fermionic model

The specific model we are interested in is that of a 2-band metal interacting under a forward-scattering interaction.

The Hamiltonian for this system in k-space is:

$$
\begin{aligned}
H &= H_{\text{kin}} + H_{\text{int}} && (2.1)\\
&= \sum_{\mathbf{k},n} \psi^{\dagger}_{n\mathbf{k}}(\varepsilon_n(|\mathbf{k}|) - \mu)\psi_{n\mathbf{k}} + \sum_{\mathbf{q}} \left(\frac{f_l(\mathbf{q})}{2}\right) \sum_{i=1,2} \sum_{\mu=x,y} \Phi_{li\mu}(\mathbf{q})\Phi_{li\mu}(-\mathbf{q}) && (2.2)
\end{aligned}
$$

where $n = 1, 2$ is a band index of an isotropic dispersion $\epsilon_{1,2}(|\mathbf{k}|)$, $f_l(\mathbf{q})$ is an interaction kernel in momentum space with units of energy and $\mathbf{q}$ is the momentum transfered. This is an effective Hamiltonian with a Hilbert space that is restricted to a shell around a common Fermi energy, μ. Hence, our ground state is that of a filled Fermi sea. The interaction comes from Fermi bilinears that forward scatter each other. This interaction is parametrized by the Fermi liquid parameter, $f_l(\mathbf{q})$. This interaction is a short-range interaction, which has screening built in already. We take the following Lorentzian form for this interaction [Oganesyan et al., 2001]

$$f_l(\mathbf{q}) = \frac{f_l(0)}{1 + \kappa_l f_l(0)|\mathbf{q}|^2} \tag{2.3}$$

where the parameter κ_l has units of $(\text{Energy})^{-1} \times (\text{Length})^2$ and characterizes the non-locality of the interaction. The other parameter $f_l(0) < 0$ controls the strength of the attractive interaction. The interaction kernel in real space is

$$\tilde{f}_l(\mathbf{r}) = \sum_{\mathbf{q}} f_l(\mathbf{q}) \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}} \tag{2.4}$$

and is an ideal δ-function for $\kappa_l = 0$ but is otherwise a smeared by $\sqrt{f_l(0)\kappa_l}$ for $\kappa_l > 0$. We will often imagine κ_l to be small but finite and approximate $\tilde{f}_l(\mathbf{r}) \approx f_l(0) L^2 \delta(\mathbf{r})$. We have only considered forward scattering and assumed that only one l-partial wave is at play. The interaction part of the Hamiltonian is quadratic in the fermion bilinears which are defined as

$$\Phi_{l,1,\mu}(q) = \sum_{k,\alpha,\beta} : \psi^\dagger_\alpha(k + q/2) \cos(l\theta_k) \sigma^{\alpha\beta}_\mu \psi_\beta(k - q/2) : \tag{2.5}$$

$$\Phi_{l,2,\mu}(q) = \sum_{k,\alpha,\beta} : \psi^\dagger_\alpha(k + q/2) \sin(l\theta_k) \sigma^{\alpha\beta}_\mu \psi_\beta(k - q/2) : \tag{2.6}$$

where $\sigma^{x,y}$ are the two of the three Pauli matrices. In most of the latter analyses we will limit ourselves with $l = 2$ channel. In this channel the interaction Hamiltonian is a quadrupolar interaction.

$$\Phi_{\mu i}(x) := \psi^\dagger_\alpha(x)\mathcal{O}^{\mu i}_{\alpha\beta}(-i\mathbf{D})\psi_\beta(x) \tag{2.7}$$

$$\mathcal{O}^{\mu i}_{\alpha\beta}(-i\mathbf{D}) := \left(\frac{\sigma^\mu_{\alpha\beta}\tau^i_{ab}}{k_F^2}\right)(-i\overleftrightarrow{D}_a)(-i\overleftrightarrow{D}_b) \quad \text{or in k-space}$$

$$\mathcal{O}^{\mu i}_{\alpha\beta}(\mathbf{k})\mid_{\mathbf{A}=0} := \left(\frac{\sigma^\mu_{\alpha\beta}\tau^i_{ab}}{k_F^2}\right)k_a k_b \tag{2.8}$$

where the matrices

$$\tau^1 \equiv \sigma^z, \tau^2 \equiv \sigma^x$$

are like the invariant tensors of the quadrupole channel viz.

$$\tau^1_{ab}p_a p_b = p_1^2 - p_2^2 \quad \text{and} \quad \tau^2_{ab}p_a p_b = 2p_a p_b. \tag{2.9}$$

The operator $\overleftrightarrow{\mathbf{D}} \equiv \overleftrightarrow{\nabla} + ie\mathbf{A} \equiv \frac{1}{2}(\overleftarrow{\nabla} - \overrightarrow{\nabla}) + ie\mathbf{A}$ is the symmetrized covariant derivative operator, which is a convenient convention for defining explicitly Hermitian velocity operators. It will be shown later that this model breaks time-reversal symmetry spontaneously when condensates form in these particle-hole channels. The order parameters then can be taken as two component vectors,

$$\vec{\Phi}_{li} = \begin{pmatrix} \Phi_{l,i,x}(0) \\ \Phi_{l,i,y}(0) \end{pmatrix}. \tag{2.10}$$

It will be important for the perturbative expansion and for going beyond the mean-field to carefully examine the dispersion expansion. Keeping higher order terms in the expansion introduces non-linear terms that relate to the curvature of the Fermi surfaces. We will start with a fermionic Hamiltonian defined in Eq. 2.2.

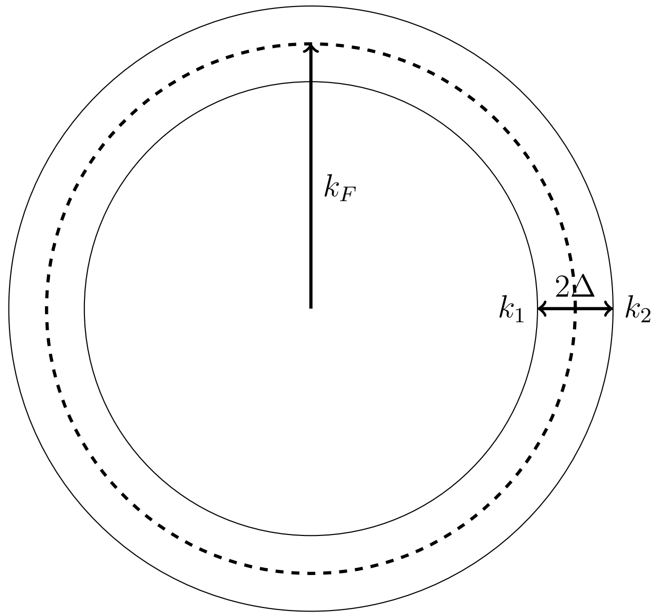


Figure 2.1: Fermi surfaces of two-band model. The momenta are measured with respect to the common Fermi momentum. The two Fermi surfaces are at a distance 2Δ apart in k-space

The dispersion relation, $\epsilon(k)$, can be Taylor-expanded as:

$$\epsilon_{1,2}(k_F+\vec{k})-\mu \approx \left.\frac{\partial\epsilon(k')}{\partial k'}\right|_{k_F}\cdot(\vec{k}\mp\vec{\Delta})+\frac{1}{2}\left.\frac{\partial^2\epsilon(k')}{\partial k'^2}\right|_{k_F}(\vec{k}\mp\vec{\Delta})^2+\frac{1}{6}\left.\frac{\partial^3\epsilon(k)}{\partial k^3}\right|_{k_F}(\vec{k}\mp\vec{\Delta})^3+..., \quad (2.11)$$

where $\{1,2\}$ are the band labels.

First let us investigate the case of linear dispersion. In this case, the kinetic piece of the Hamiltonian becomes:

$$H_{kin} = \sum_k \vec{v}_F\cdot\vec{k}\left(\psi_1^\dagger(k)\psi_1(k)+\psi_2^\dagger(k)\psi_2(k)\right)+\frac{\vec{v}_F\cdot\vec{\Delta}}{2}\sum_k\left(\psi_2^\dagger(k)\psi_2(k)-\psi_1^\dagger(k)\psi_1(k)\right). \quad (2.12)$$

In the above equation, the momenta of the top and the bottom bands are split by 2Δ, as illustrated in Figure 2.1. This simplifies our normal-ordering procedure to be with respect to the common Fermi momentum. We will show later that when the quadratic terms of the total Hamiltonian ($H_{kin}+H_{int}$) is not positive definite, the higher order terms in the dispersion expansion have to be included, due to stability. Our approach in treating the non-linear dispersion effects is similar to work done in [Barci and Oxman, 2003]. These two-band fermionic fluids can be treated by using Hartree-Fock mean-field theory. This was done by Kai Sun and Eduardo Fradkin in 2008[Sun and Fradkin, 2008]. In the next section, I will summarize the results obtained using this framework.

2.2 Mean-field theory treatment

This work builds off of the result from [Sun and Fradkin, 2008] that one type of the ground state of fluids may exhibit a spontaneous anomalous Hall effect at zero magnetic field. This is also shown to be related to the Berry connection in the mean-field limit. Hartree-Fock theory treats the model as an extension of a Landau-Fermi liquid. This means that the systems have well-defined quasi-particle excitations. This assumption also implies that within this weak-coupling limit (close to the Fermi liquid fixed point), the system with N Fermi surfaces would obtain an emergent $U(1) \otimes ... \otimes U(1)$ symmetry. In this mean-field approach, one can characterize the time-reversal symmetry breaking phases by examining how the one-particle states and the effective Fermi surfaces transform under the following symmetries: time-reversal (**T**), chiral (**C**), and the space inversion which is equal to parity in two dimensions(**P**).

There are two types of T-reversal breaking phases: Type I phase which breaks **T** and **P** symmetries but preserves C and the combined IT symmetries; Type II breaks **T** and **C** symmetries but preserves **I** and the combined **CT** symmetries.

Type II phase exhibits non-quantized anomalous Hall effect in the absence of external magnetic fields which corresponds to the Berry curvature on the Fermi surface while Type I does not. From basic dimensional and symmetry considerations, we can define the Landau free energy that preserves the spatial symmetry at the internal $U(1)$ symmetry:

$$F = m(|\vec{\phi}_{l1}|^2 + |\vec{\phi}_{l2}|^2) + u(|\vec{\phi}_{l1}|^2 + |\vec{\phi}_{l2}|^2)^2 + 4v(\vec{\phi}_{l1} \times \vec{\phi}_{l2})^2 + \text{higher order terms}, \quad (2.13)$$

where the coefficients of the free energy can be determined by the standard diagramatic expansion near the critical point. They are:

$$\begin{aligned} m &= -\left(\frac{N(0)}{4} + \frac{1}{2f_l(0)}\right) + \Delta^2 \frac{N(0)}{96}\left[3\left(\frac{N'(0)}{N(0)}\right)^2 - \frac{N''(0)}{N(0)}\right], \\ u &= \frac{N(0)}{64}\left[2\left(\frac{N'(0)}{N(0)}\right)^2 - \frac{N''(0)}{N(0)}\right], \quad v = \frac{N''(0)}{48}. \end{aligned} \quad (2.14)$$

This free energy has a similar form to the spin Pomeranchuk instability state,[Wu et al., 2007] except that the symmetry is reduced from $SU(2)$ to $U(1) \otimes U(1)$. The phase diagram, illustrated in Figure 2.2, can be explored by simply tuning the parameters in the free energy.

The system has three phases: the normal phase, the α_1 phase in which $\vec{\phi}_{l1} \times \vec{\phi}_{l2} = 0$, and the phase in which $\vec{\phi}_{l1} \cdot \vec{\phi}_{l2} = 0$ and $|\vec{\phi}_{l1}| = |\vec{\phi}_{l2}|$

The boundaries between the normal phase and the $\beta_1$and α_1 phases are: $u > 0, u + v > 0$, and $m = 0$[Sun and Fradkin, 2008]. Whether the phase is transformed into the orthogonal phase (α_1) or parallel phase (β_1) depends on the sign of the v. It transforms into β_1 if $v < 0$, and into α_1 if $v > 0$. This is to be expected based on the stability of the free energy. We later derived a similar condition from the bosonized theory. Fermi surfaces of $\beta_1$and α_1 are shown in Figure 2.3.

In the α_1 phase, the two Fermi surfaces are distorted and are $\pi/2$ with respect to each other. In other words, their rotational symmetries are reduced to $2l$-discrete group. Further analysis shows that it preserves the time-reversal symmetry. To be more precise, time-reversal maps one band onto the other and vice versa. The β_1 phase has no distortion but the relative phase of the two bands (difference in the order parameter) remain locked to each

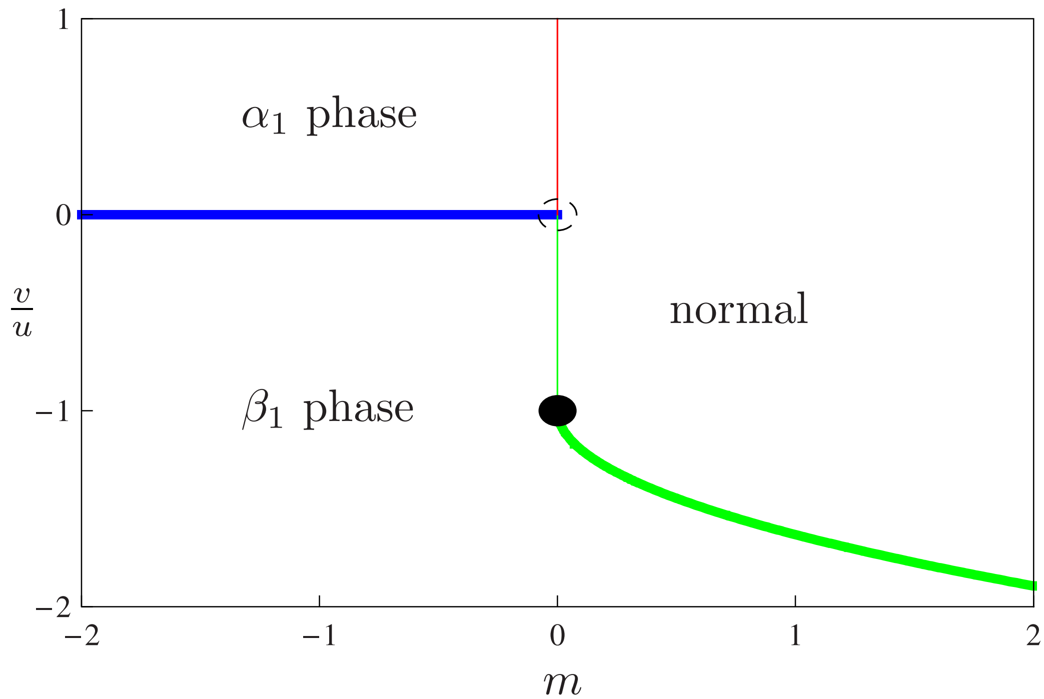


Figure 2.2: The phase diagram with the relative $U(1)$ symmetry. The thick lines are the firs order phase boundaries. The lighter lines are the second-order ones. The point where the second order and the first order phase boundaries meet is the tricritical point[Sun and Fradkin, 2008].

other but wind around the Fermi surfaces. Under time-reversal operation, the x-component remains the same but the y-component is reflected about the plane. This is illustrates in Figure 2.4. We can conclude that the α_1 phase does not break the time-reversal symmetry since there is a freedom to rotate the order parameters to undo the effect. On the other hand, the β_1 phase breaks time-reversal symmetry non-trivially. The β_1 phase breaks the chiral and time-reversal symmetry but preserves the combination. This β_1 phase is of Type II, since it exhibits a non-zero anomalous Hall conductance. The conductance can be calculated by:

$$\sigma_{xy} = \frac{n_z(1-n_z^2)}{4} \oint_{FS} \frac{dk}{2\pi} \cdot (\tilde{n}_x \vec{\nabla}_k \tilde{n}_y - \tilde{n}_y \vec{\nabla}_k \tilde{n}_x), \tag{2.15}$$

where $\tilde{n}_x = \frac{n_x}{\sqrt{1-n_z^2}}$ and $\tilde{n}_y = \frac{n_y}{\sqrt{1-n_z^2}}$. The n vectors are defined in the 2x2 single-particle Hamiltonian decomposed into $\{I, \sigma^\mu\}$ basis where the σ^μ are the Pauli matrices. Note

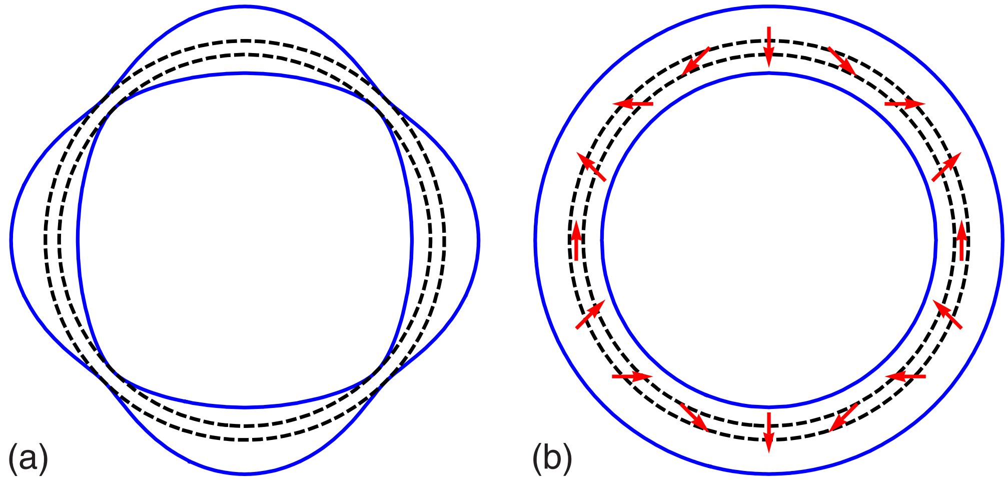


Figure 2.3: These figures show the Fermi surfaces of the β_1 and α_1 phases. (a) α_1 is characterized by the distortion of the Fermi surface. (b) β_1 phase has no distortion but the relative phase of the two band (difference in the order parameter) remain locked to each other but wind around the Fermi surfaces.[Sun and Fradkin, 2008].

that the integral traces over the Fermi surface. This treatment here is equivalent to Haldane's treatment of the anomalous Hall conductance as a Berry curvature on the Fermi surface[Haldane, 2004]. Looking at the prefactor, $\frac{n_z(1-n_z^2)}{4}$, we can conclude that the conductance is unquantized.

In summary, we obtain Landau free energy using the Hartree-Fock treatment. We also found that the coefficients of the free energy depends on the curvature of the Fermi surfaces. By adjusting the parameters we could explore different regions in the phase diagram. Furthermore the $\mathbf{T}$ symmetry breaking phase β_1 is shown to have non-trivial winding of the relative phase factor around the Fermi surface. It also has an non-vanishing anomalous Hall conductance which can be related to a Berry curvature on the Fermi surface. However this phenomenological construction using mean-field theory has its shortcomings. For example the perturbative expansion with respect to the Fermi liquid fixed point is no longer a good description for strongly-correlated systems. In the rest of this paper, I will attempt

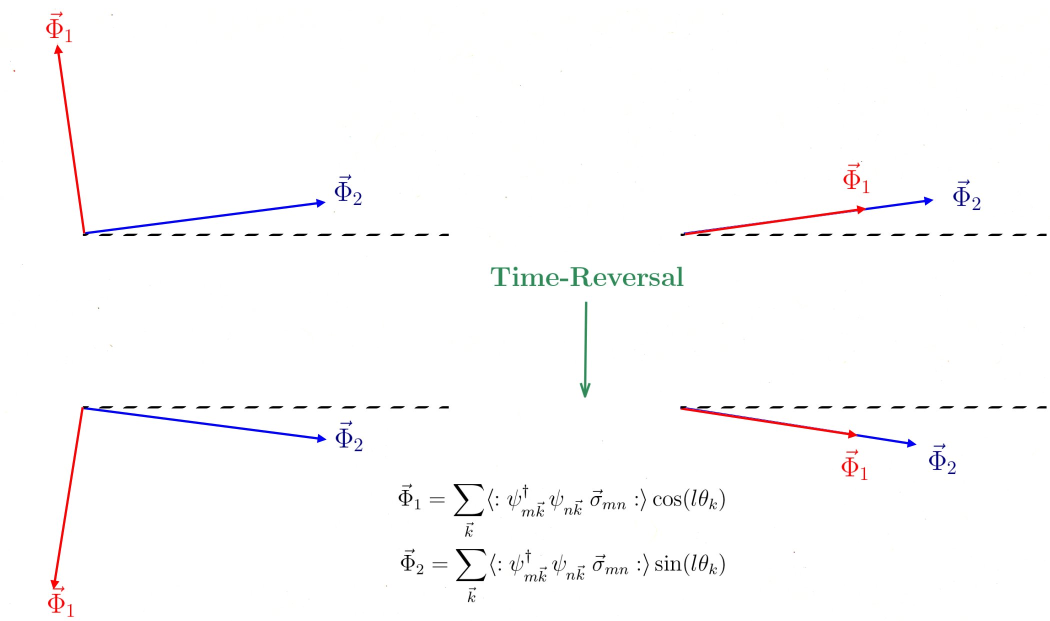


Figure 2.4: representation of the order parameters as a two-component vector. The time-reversed version is also illustrated. One can see that only one of the case breaks time-reversal symmetry.

to develop a non-perturbative theoretical framework which employs the technique of higher dimensional bosonization and digrammatic methods.

Chapter 3

Multidimensional Bosonization Approach

In this chapter the two-band metallic model presented earlier is studied using the technique of multidimensional bosonization. Bosonized dictionary for mapping the microscopic fermionic theory to the coarse-grained(patched) bosonic theory is established. The approach taken is to make use of the property that at the weak-interaction limit, the band-fermions are separately conserved. Therefore the band fermions can be bosonized independently. The interacting nature is introduced by applying the bosonization dictionary directly to the interaction term.

Since in higher dimension bosonization is significantly different from the 1-D one, in which bosonization is exact([Houghton et al., 2000], [Kwon et al., 1995]), first section presents the derivation of two-dimensional bosonization. In the next section, the technology of bosonization for arbitrary order of gradient terms is developed explicitly. This is important because to capture the curvature effect correctly and for stability reason, the dispersion relation has to be expanded to third order. We then derive bosonized action of the two-band metal. Euler-Lagrange equations of motion follow directly from the action. Solution of the β_1 phase is computed. Furthermore a first order correction from the interaction to the Pomerunchuk critical point can be acheived in an elegant manner in the bosonized framework.

3.1 Multi-Dimensional Bosonization

As mentioned earlier bosonization is exact in one-dimensional Luttinger liquid. To expand the technique to higher dimension required a careful analysis of the current algebra. I use the

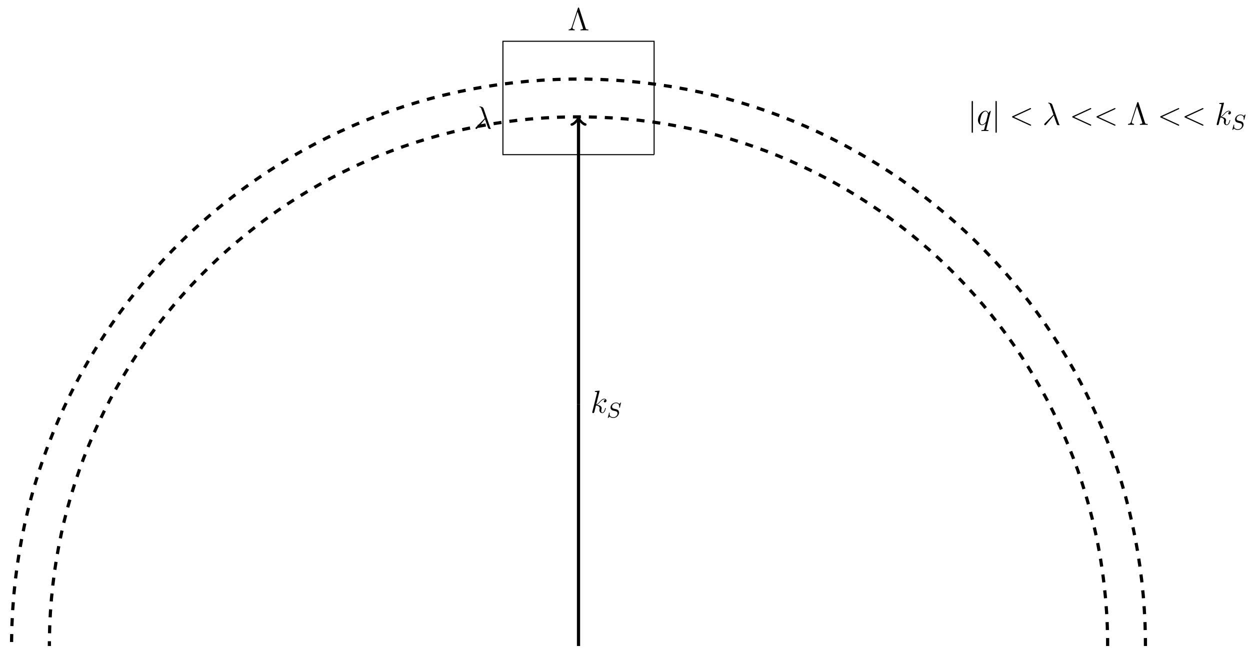


Figure 3.1: Illustration of the tiling procedure for the Fermi surface in 2 dimensions. The Hilbert space is restricted to the modes in the box centered at k_S with the tiling width, Λ referring to the patch S. λ is the shell thickness which plays the role of the momentum cut-off. Later, to get to the continuous limit, we take $\Lambda \to 0$ and $N \to \infty$ simultaneously.

method of multi-dimensional bosonization to study this problem. The spectrum of the free-fermion spectrum can be reproduced by considering the particle-hole excited state in the radial direction from the Fermi surface. Therefore, it is possible to think of each radial direction as a 1-D problem. This idea was first implemented by Luther[Luther, 1976]. The drawback is that each radial direction cannot interact with each other. Haldane [Haldane, 2005] proposed that by adopting a different geometry, one could incorporate the coupling between different bosonic degrees of freedoms. The idea is illustrated in Figure 3.1: the Fermi surface is coarse-grained into different patches that are centered at the Fermi momentum, k_S rather than along radial rays. This approach is adopted by Marston and Houghton and collaborators, who emphasize the use of the current algebras.[Houghton and Marston, 1993][Houghton et al., 2000] An independent work was done by Castro Neto and Fradkin using the

coherent-state path integral and considering the particle-hole pair as the fluctuation of quantum membrane.[Castro Neto and Fradkin, 1994b][Castro Neto and Fradkin, 1994a] Basic properties of the Landau Fermi liquid can be deduced in this bosonized language.

Using the patched construction The fermion annihilation operator in k-space, $\psi(x) = \sum_k \frac{\psi_k e^{ik\cdot x}}{L}$, can be expanded into operators in different patches as:

$$\begin{aligned} \psi(x) &= \sum_S \sum_{q\in S} \psi_{k_S+q} \frac{e^{i(k_S+q)\cdot x}}{L} \\ &= \sum_S \psi_{S,n}(x) \frac{e^{ik_S\cdot x}}{\sqrt{N}}, \end{aligned} \tag{3.1}$$

where S is a rectangular box in momentum space centered at k_s with dimension Λxλ, as illustrated in Figure 3.1. We can think of the procedure as separating the operator into the slow and fast modes, and then integrating out the fast modes. This operator obeys the canonical anti-commutation relation if this operator corresponds to the same patch:

$$\{\psi_S(x), \psi_T^\dagger(y)\} = \delta_{s,T}\delta^2(x-y). \tag{3.2}$$

3.1.1 Simple Bosonization

Details of the bosonization procedure are described here. Starting with the patched linear Hamiltonian where a previous smearing and coarse-graining has taken place to isolate the low energy physics at Fermi-energy

$$H = \sum_S \int d^2x\, \psi_S^\dagger(\vec{x}) \left[-i\hbar v_F(\vec{n}_S\cdot\vec{\nabla})\right] \psi_S(\vec{x}). \tag{3.3}$$

The patched fermions satisfy

$$\{\psi_S(\vec{x}), \psi_T^\dagger(\vec{x}')\} = \delta_{ST}\delta_S(\vec{x}-\vec{x}') \tag{3.4}$$

$$\delta_S(\vec{x}-\vec{x}') = \frac{1}{L^2}\sum_{\vec{k}\in S} \mathrm{e}^{i(\vec{k}-\vec{k}_S)\cdot(\vec{x}-\vec{x}')} \tag{3.5}$$

where a sufficiently fine grid of the Fermi-surface shell has been constructed such that the RHS of the anti-commutator is a sharp δ-like function. The system size or area is L^2 and $\vec{k}_S$ is a Fermi-vector centered on the S-th patch. All patches are identical with the dimensions shown in the schematic of Fig. 3.2. Next the $\delta-$function is decomposed into normal and

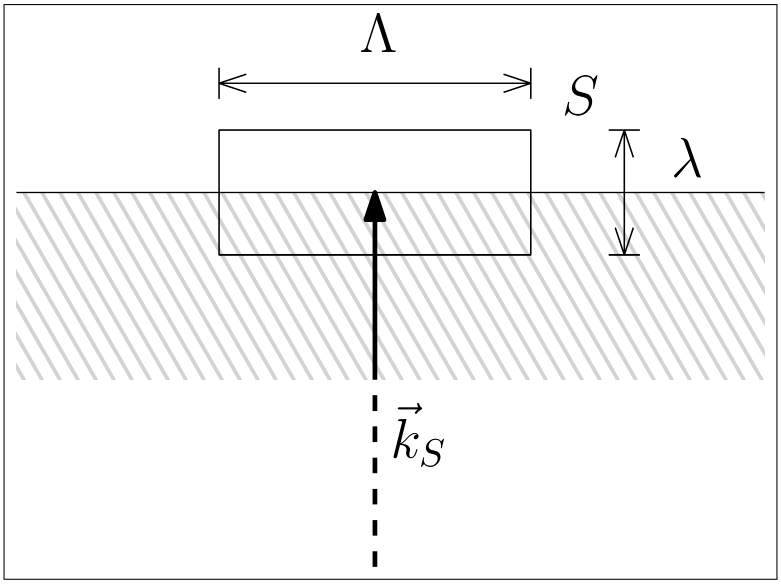


Figure 3.2: Schematic of a patch around the two dimensional Fermi-surface of radius k_F. The hatched region denotes the filled Fermi sea. λ represents a momentum shell cutoff and $\Lambda \ll k_F$ represents the "lateral" bandwidth. The total number of k-points enclosed in a box is $(\Lambda\lambda L^2)/(2\pi)^2$.

tangential components,

$$\delta_S(\vec{x}-\vec{y}) = \underbrace{\delta_n(\vec{n}_S \cdot [\vec{x}-\vec{y}])}_{\text{normal}} \underbrace{\delta_t(\vec{t}_S \cdot [\vec{x}-\vec{y}])}_{\text{tangential}} \tag{3.6}$$

where

$$
\begin{aligned}
\delta_S(\vec{r}) &= \frac{1}{L^2} \sum_{\vec{q}+\mathbf{k}_S \in S} \mathrm{e}^{i\mathbf{q}\cdot[r_n \vec{n}_S + r_t \vec{t}_S]} \\
&= \frac{1}{L^2} \sum_{\mathbf{q}+\mathbf{k}_S \in S} \mathrm{e}^{i[\mathbf{q}\cdot\vec{n}_S] r_n}\ \mathrm{e}^{i[\mathbf{q}\cdot\vec{t}_S] r_t} \\
&= \left(\frac{1}{L} \sum_{q_n} \mathrm{e}^{i q_n r_n} \right) \left(\frac{1}{L} \sum_{q_t} \mathrm{e}^{i q_t r_t} \right) \\
&=: \delta_n(\vec{n}_S \cdot \mathbf{r})\ \delta_t(\vec{t}_S \cdot \mathbf{r})
\end{aligned} \tag{3.7}
$$

also $\{k_n = k_S + q_n,\ k_t = q_t\}$ or $\{q_n = k_n - k_S,\ q_t = k_t\}$. Hence

$$
\delta_S(\mathbf{r}) = \underbrace{\left(\frac{1}{L} \sum_{\mathbf{k}\in S} \mathrm{e}^{i[k_n - k_S] r_n} \right)}_{\delta_n(r_n)} \underbrace{\left(\frac{1}{L} \sum_{\mathbf{k}\in S} \mathrm{e}^{i k_t r_t} \right)}_{\delta_t(r_t)} \tag{3.8}
$$

Define the density operator as

$$
\begin{aligned}
\delta n_S(\mathbf{x}) &\equiv :\psi_S^\dagger(\mathbf{x})\psi_S(\mathbf{x}): \qquad \text{normal order and regulate by point-splitting} \\
&= \lim_{\epsilon\to 0} \left[\psi_S^\dagger(\mathbf{x} + \tfrac{\epsilon}{2}\vec{n}_S)\ \psi_S(\mathbf{x} - \tfrac{\epsilon}{2}\vec{n}_S) - \langle G|\psi_S^\dagger(\mathbf{x} + \tfrac{\epsilon}{2}\vec{n}_S)\ \psi_S(\mathbf{x} - \tfrac{\epsilon}{2}\vec{n}_S)|G\rangle \right]
\end{aligned} \tag{3.9}
$$

$|G\rangle$ is the reference vacuum defined by the filled Fermi sea in Fig. 3.2

$$
|G\rangle := \prod_S \overleftarrow{\prod_{\{\mathbf{q}+\mathbf{k}_S \in S | q_n < 0\}}} \psi^\dagger_{\mathbf{q}+\mathbf{k}_S} |0\rangle \tag{3.10}
$$

where the operator product $\overleftarrow{\prod}$ denotes an ordered product of creation operators. One then needs to then calculate the point-split average density. In momentum space

$$
\psi_S(\mathbf{x}) = \frac{1}{L} \sum_{\mathbf{k}\in S} \psi_{\mathbf{k}} \mathrm{e}^{i[\mathbf{k}-\mathbf{k}_S]\cdot\mathbf{x}}, \quad \psi_S^\dagger(\mathbf{x}) = \frac{1}{L} \sum_{\mathbf{k}\in S} \psi_{\mathbf{k}}^\dagger \mathrm{e}^{-i[\mathbf{k}-\mathbf{k}_S]\cdot\mathbf{x}} \tag{3.11}
$$

which gives for $\vec{\epsilon} = \epsilon \vec{n}_S$

$$\begin{aligned}\psi_S^\dagger(\mathbf{x}+\tfrac{\vec{\epsilon}}{2})\psi_S(\mathbf{x}-\tfrac{\vec{\epsilon}}{2}) &= \frac{1}{L^2}\sum_{\mathbf{k}_1,\mathbf{k}_2\in S}\psi_{\mathbf{k}_1}^\dagger\psi_{\mathbf{k}_2}\mathrm{e}^{-i[\mathbf{k}_1-\mathbf{k}_S]\cdot[\mathbf{x}+\vec{\epsilon}/2]+i[\mathbf{k}_2-\mathbf{k}_S]\cdot[\mathbf{x}-\vec{\epsilon}/2]} \\ &= \frac{1}{L^2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\psi_{\mathbf{k}_1}^\dagger\psi_{\mathbf{k}_2}\mathrm{e}^{-i[\mathbf{k}_1-\mathbf{k}_2]\cdot\mathbf{x}-i[\mathbf{k}_1+\mathbf{k}_2]\cdot\vec{\epsilon}/2+i\mathbf{k}_S\cdot\epsilon}\end{aligned} \tag{3.12}$$

Naively, if one was really calculating the density

$$\begin{aligned}\langle G|\psi_S^\dagger(\mathbf{x}+\tfrac{\vec{\epsilon}}{2})\psi_S(\mathbf{x}-\tfrac{\vec{\epsilon}}{2})|G\rangle &= \frac{1}{L^2}\sum_{\mathbf{k}\cdot\vec{n}_S<0} 1\cdot e^{-i[\mathbf{k}-\mathbf{k}_S]\cdot\vec{\epsilon}} \\ &= \underbrace{\left(\frac{1}{L}\sum_{q_n<0}\mathrm{e}^{-iq_n\epsilon}\right)}_{\frac{1}{L}\left[\frac{1-e^{+i(\lambda\epsilon/2)}}{1-e^{+i(2\pi\epsilon/L)}}\right]}\underbrace{\left(\frac{1}{L}\sum_{q_t}1\right)}_{\delta_t(0)=\frac{\Lambda}{2\pi}}\end{aligned} \tag{3.13}$$

then taking the limit $\epsilon \to 0^+$ and using L'Hopital's rule on the first factor produces

$$\langle G|\psi_S^\dagger(\mathbf{x}+0^+\tfrac{\vec{n}_S}{2})\psi_S(\mathbf{x}-0^+\tfrac{\vec{n}_S}{2})|G\rangle = \frac{\lambda}{4\pi}\times\frac{\Lambda}{2\pi} = \frac{1}{2}\left[\frac{\lambda\Lambda}{(2\pi)^2}\right] \tag{3.14}$$

which is basically $1/2$ the density of k-points in the patch S. But this is incorrect because one is not interested in calculating the density but the value of the near equal-time point-split propagator which takes a nominally divergent imaginary value. To achieve this, one must explicitly regulate the spatial infinity limits by $x + i0^+$ or $\epsilon + i0^+$ which instead removes an exponential term from the geometric sum to yield instead

$$
\begin{aligned}
\langle G|\psi_S^\dagger(\mathbf{x}+\tfrac{\vec{\epsilon}}{2})\psi_S(\mathbf{x}-\tfrac{\vec{\epsilon}}{2})|G\rangle &= \underbrace{\left(\frac{1}{L}\sum_{q_n<0}\mathrm{e}^{-iq_n\epsilon}\right)}_{\frac{1}{L}\left[\frac{1}{1-e^{+i(2\pi\epsilon/L)}}\right]}\underbrace{\left(\frac{1}{L}\sum_{q_t}1\right)}_{\delta_t(0)=\frac{\Lambda}{2\pi}} \\
&= \left(\frac{i}{2\pi\epsilon}\right)\left(\frac{\Lambda}{2\pi}\right) \\
&= \frac{i\lambda\Lambda}{(2\pi)^2}
\end{aligned}
\tag{3.15}
$$

with $\epsilon = \lambda^{-1}$ such that the short-distance ϵ is set by the momentum cutoff. Likewise

$$
\psi_S^\dagger(\mathbf{x}+0^+\vec{n}_S)\psi_S(\mathbf{x}-0^+\vec{n}_S) = \frac{1}{L^2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\psi_{\mathbf{k}_1}^\dagger\psi_{\mathbf{k}_2}\mathrm{e}^{-i[\mathbf{k}_1-\mathbf{k}_2]\cdot\mathbf{x}} \tag{3.16}
$$

giving the density or occupation differential as

$$
\delta n_S(\mathbf{x}) = \frac{1}{L^2}\sum_{\mathbf{k}_1,\mathbf{k}_2\in S}\psi_{\mathbf{k}_1}^\dagger\psi_{\mathbf{k}_2}\mathrm{e}^{-i[\mathbf{k}_1-\mathbf{k}_2]\cdot\mathbf{x}} + i\left[\frac{\lambda\Lambda}{(2\pi)^2}\right] \tag{3.17}
$$

Next rewrite $S = \mathbf{k}_S + \underbrace{\square}_{\text{box}}$ with $\mathbf{k}_i = \mathbf{k}_S + \mathbf{q}_i,\ i = 1, 2$ where $\mathbf{q}_i \in \square$.

$$\delta n_S(\mathbf{x}) = \frac{1}{L^2} \sum_{\mathbf{q}_1, \mathbf{q}_2 \in \square} \psi^\dagger_{\mathbf{k}_S + \mathbf{q}_1} \psi_{\mathbf{k}_S + \mathbf{q}_2} \mathrm{e}^{-i[\mathbf{q}_1 - \mathbf{q}_2] \cdot \mathbf{x}} + i \left[\frac{\lambda \Lambda}{(2\pi)^2} \right]$$

(express in relative momentum $\mathbf{q} = \mathbf{q}_2 - \mathbf{q}_1$)

$$= \frac{1}{L^2} \sum_{\mathbf{q} \in \square + \square} \underbrace{\left(\sum_{\mathbf{q}_1 \in \square} \psi^\dagger_{\mathbf{k}_S + \mathbf{q}_1} \psi_{\mathbf{k}_S + \mathbf{q}_1 + \mathbf{q}} \right)}_{=: \rho_{S(-\mathbf{q})}} \mathrm{e}^{i \mathbf{q} \cdot \mathbf{x}} + i \left[\frac{\lambda \Lambda}{(2\pi)^2} \right] \tag{3.18}$$

Density operator but

S dependent and chiral.

$$\delta n_S(\mathbf{x}) = \frac{1}{L^2} \sum_{\mathbf{q} \in \square + \square} \rho_{S(-\mathbf{q})} \ \mathrm{e}^{i \mathbf{q} \cdot \mathbf{x}} + i \left[\frac{\lambda \Lambda}{(2\pi)^2} \right] \tag{3.19}$$

It is natural to include the $\mathbf{q} = 0$ mode into the definition of ρ_S

$$\rho_{S(-\mathbf{q})} := \sum_{\mathbf{q}_1 \in \square} \left(\psi^\dagger_{\mathbf{k}_S + \mathbf{q}_1} \psi_{\mathbf{k}_S + \mathbf{q}_1 + \mathbf{q}} + i \left[\frac{(2\pi)^2}{\lambda \Lambda L^2} \right] \delta_{\mathbf{q}, 0} \right), \qquad \sum_{\mathbf{q} \in \square} 1 = \frac{\lambda \Lambda L^2}{(2\pi)^2} \tag{3.20}$$

Regardless one has

$$\delta n_S(\mathbf{x}) = \frac{1}{L^2} \sum_{\mathbf{q} \in \square + \square} \rho_{S(-\mathbf{q})} \ \mathrm{e}^{i \mathbf{q} \cdot \mathbf{x}} - \ \text{(some constant)} \tag{3.21}$$

and

$$[\delta n_S(\mathbf{x}), \delta n_T(\mathbf{x})] = \frac{\delta_{S,T}}{L^2} \sum_{\substack{\mathbf{q}_1,\mathbf{q}_2 \\ \mathbf{q}_1,\mathbf{q}_2 \neq 0}} \underbrace{[\rho_{S\mathbf{q}_1}, \rho_{S\mathbf{q}_2}]}_{\substack{\text{satisfies current} \\ \text{algebra}}} \ \mathrm{e}^{-i\mathbf{q}_1 \cdot \mathbf{x} - \text{ß}\mathbf{q}_2 \cdot \vec{y}} \tag{3.22}$$

$$[\rho_{S\mathbf{q}}, \rho_{S\vec{q}'}] = -N_S \ \underbrace{(\vec{n}_S \cdot \mathbf{q})}_{\substack{\text{geometric} \\ \text{increase} \\ \text{in occupation} \\ \text{number}}} \ L \ \delta_{\mathbf{q},-\vec{q}'} \tag{3.23}$$

where N_S is some constant that is to be determined by choice of normalization and cutoff. Substituting

$$\begin{aligned} [\delta n_S(\mathbf{x}), \delta n_T(\vec{y})] &= -\left(\frac{\delta_{S,T}}{L}\right) N_S \sum_{\mathbf{q} \in \square + \square} (\vec{n}_S \cdot \mathbf{q}) \ \mathrm{e}^{-i\mathbf{q} \cdot [\mathbf{x} - \vec{y}]} \\ &= -i \left(\frac{\delta_{S,T} N_S}{L}\right) \sum_{\mathbf{q}} (\vec{n}_S \cdot \vec{\nabla}_x) \mathrm{e}^{-i\mathbf{q} \cdot [\mathbf{x} - \vec{y}]} \\ &= -i \left(\frac{\delta_{S,T} N_S}{L}\right) (\vec{n}_S \cdot \vec{\nabla}) \delta(\mathbf{x} - \vec{y}). \end{aligned} \tag{3.24}$$

The δ-function above is really regulated by the cutoffs Λ, λ of $\square$. By dimensional analysis $[N_S]$ carries no dimension. Comparing with the convention from Ref. [Lawler et al., 2006] it appears that $N_S = +i\hbar v_F N(0) L$ which is dimensionless. This yields

$$[\delta n_S(\mathbf{x}), \delta n_T(\mathbf{x})] = -i\hbar v_F N(0) (\vec{n}_S \cdot \vec{\nabla}) \delta(\mathbf{x} - \vec{y}) \tag{3.25}$$

which has a Schwinger term[1] appropriate for right-movers on the right hand side. Note that the density of states $N(0)$ factor is a 2D density of states at the Fermi-energy. Recall that the total number of states in $\square$ is implied by

$$L^2 \times N(0)\hbar v_F \lambda = \frac{\Lambda\lambda}{(2\pi)^2} L^2 \tag{3.26}$$

or

$$\hbar v_F N(0) = \frac{\Lambda}{(2\pi)^2} \tag{3.27}$$

and

$$\hbar v_F N(0) L = \frac{\Lambda L}{(2\pi)^2} = \frac{\Lambda}{2\pi/L}\frac{1}{2\pi} = \frac{1}{2\pi} \times (\text{number of tangential states}) \tag{3.28}$$

Therefore

$$N_S = \left(\frac{\Lambda}{2\pi/L}\right) \times \frac{1}{2\pi} \tag{3.29}$$

which implies a 1D density algebra but multiplied by a $\frac{\Lambda}{2\pi/L}$ factor. Note that the negative sign is symptomatic of right-movers. This is a separate interpretation of the density or Kac-Moody algebra where $\delta n_S(\mathbf{x})$ is really analogous to the the chiral right-moving current $j_R(x)$ in 1D. Comparing with the 1D bosonization rules and adopting the convention of [Lawler et al., 2006], chiral bosonic fields ϕ_S are introduced according to the following relation with the patch fermions,

$$\boxed{\psi_S(\mathbf{x}) = \eta_S(x_t)\sqrt{\hbar N(0) v_F \lambda} : \mathrm{e}^{-i\phi_S(\mathbf{x})} :} \tag{3.30}$$

[1] In general there is no easy way to determine the form of Schwinger term from a classical current algebra. It varies from microscopic theory to microscopic theory.

where $\hbar v_F N(0)\lambda = \Lambda\lambda/(2\pi)^2$ such that the constant of normalization agrees. $\eta_S(x_t)$ is a Klein factor which depends on the patch and transverse position. Identifying ϕ_S with a right-moving bosonic field in the $\vec{n}_S$ direction yields the following commutation relations

$$\begin{aligned}[\phi_S(\mathbf{x}), \phi_T(\vec{y})] &= 4\pi \cdot \left(\frac{-i}{4}\right) \mathrm{sign}(\vec{n}_S \cdot [\mathbf{x} - \vec{y}]) \cdot \delta_{S,T} \cdot \delta_t(\vec{t}_S \cdot [\mathbf{x} - \vec{y}]) \left(\frac{2\pi}{\Lambda}\right) \\ &= -i \left(\frac{2\pi}{\Lambda}\right) \pi \ \delta_{S,T} \ \mathrm{sign}(\vec{n}_S \cdot [\mathbf{x} - \vec{y}]) \ \delta_t(\vec{t}_S \cdot [\mathbf{x} - \vec{y}]) \end{aligned} \tag{3.31}$$

or

$$[\phi_S(x_n \vec{n}_S + x_t \vec{t}_S), \phi_S(x'_n \vec{n}_S + x_t \vec{t}_S)] = -i\pi \ \mathrm{sign}(x_n - x'_n). \tag{3.32}$$

As a check one can compute the expression for the current

$$\delta n_S(\mathbf{x}) = \lim_{\epsilon \to 0^+} \left[\psi_S^\dagger(\mathbf{x} + \vec{\epsilon}) \ \psi_S(\mathbf{x} - \vec{\epsilon}) - \langle \psi_S^\dagger(\mathbf{x} + \vec{\epsilon}) \ \psi_S(\mathbf{x} - \vec{\epsilon}) \rangle \right]. \tag{3.33}$$

Recall the non-interacting propagator for free relativistic fermions ψ_S

$$\langle T_t \psi_S(\mathbf{x}, t) \psi_T^\dagger(0, 0) \rangle = \frac{i\Theta(t) \ \delta_t(\vec{n}_s \cdot \mathbf{x}) \ \delta_{S,T}}{2\pi(\vec{n}_S \cdot \mathbf{x} - v_F t + i0^+)} + \frac{i\Theta(-t) \ \delta_t(\vec{n}_S \cdot \mathbf{x}) \ \delta_{S,T}}{2\pi(\vec{n}_S \cdot \mathbf{x} - v_F t - i0^+)} \tag{3.34}$$

Then using $\delta_t(0) = \Lambda/(2\pi)$ one computes the point-split density to be

$$\langle \psi_S^\dagger(\mathbf{x} + \vec{\epsilon}, 0) \psi_S(\mathbf{x}, 0) \rangle = \left(\frac{i}{2\pi\epsilon}\right) \frac{\Lambda}{2\pi} = \frac{i\Lambda\lambda}{(2\pi)^2}. \tag{3.35}$$

The shortest scale in x_n is always taken to be $\epsilon = 1/\lambda$ to give the correct normalization. To get this from the bosonization rules, one begins with

$$\psi_S^\dagger(\mathbf{x} + \vec{\epsilon}) \psi_S(\mathbf{x}) = \left(\frac{\Lambda\lambda}{(2\pi)^2}\right) : \mathrm{e}^{i\phi_S(\mathbf{x} + \vec{\epsilon})} :: \mathrm{e}^{-i\phi_S(\mathbf{x})} : \tag{3.36}$$

Next the following all important operator relation for single bosonic operators A, B is used

$$\boxed{: \mathrm{e}^{A} :: e^{B} : \quad \equiv \quad : \mathrm{e}^{A+B} : \mathrm{e}^{\langle AB \rangle}} \tag{3.37}$$

where the $\langle \star \rangle$ is taken with respect to the bosonic number vacuum. Thus

$$\psi_S^\dagger(\mathbf{x}+\vec{\epsilon})\psi_S(\mathbf{x}) = \frac{\Lambda\lambda}{(2\pi)^2} : \mathrm{e}^{i\phi_S(\mathbf{x}+\vec{\epsilon})-i\phi_S(\mathbf{x})} : \mathrm{e}^{\langle \phi_S(\mathbf{x}+\vec{\epsilon})\phi_S(\mathbf{x})\rangle} \tag{3.38}$$

Next one needs the bosonic propagator. This can be inferred from the definition of ϕ_S in terms of density operators. To achieve the correct normalization this happens to be

$$\langle \phi_S(\mathbf{x},t)\phi_S(0,0)\rangle = -\ln|(-i\lambda)\vec{n}_S \cdot [\mathbf{x} - v_F t]| \left(\frac{\delta_t(\vec{t}_S \cdot \mathbf{x})}{\delta_t(0)}\right). \tag{3.39}$$

Thus

$$\begin{aligned} \psi_S^\dagger(\mathbf{x}+\vec{\epsilon})\psi_S(\mathbf{x}) &= \frac{\Lambda\lambda}{(2\pi)^2}\left(\frac{i}{\lambda\epsilon}\right)(1 + i : [\phi_S(\mathbf{x}+\vec{\epsilon}) - \phi_S(\mathbf{x})] :) \\ &= \frac{i\Lambda\lambda}{(2\pi)^2} - \left(\frac{\Lambda}{(2\pi)^2}\right) : \vec{n}_S \cdot \vec{\nabla}\phi_S(\mathbf{x}) : \end{aligned} \tag{3.40}$$

Therefore

$$\delta n_S(\mathbf{x}) = -\hbar v_F N(0)\, \vec{n}_S \cdot \vec{\nabla}\phi_S(\mathbf{x}) \tag{3.41}$$

as the bosonization identity for this convention. In determining this expression, it is recognized through normal ordering that the singular terms cancel in the limit $\epsilon \to 0^+$.

A simple and dirty rule to convert between 1D right-movers and patched bosons for 2D

Fermi-surfaces is given by the following table.[2]

$$
\begin{array}{|ccc|}
\hline
\psi_S(\mathbf{x}) & \leftrightsquigarrow & \left(\frac{\Lambda}{2\pi}\right)^{1/2}\psi_R(x_n)\eta_S(x_t) \\
 & & \\
\vec{n}_S \cdot \vec{\nabla} & \leftrightsquigarrow & \partial_{x_n} \\
 & & \\
\phi_S(\mathbf{x}) & \leftrightsquigarrow & -\sqrt{4\pi}\phi_R(x_n) \\
 & & \\
\frac{1}{\lambda} & \leftrightsquigarrow & a \\
 & & \\
\left(\frac{\lambda}{2\pi}\right)^{1/2} & \leftrightsquigarrow & \frac{1}{\sqrt{2\pi a}} \\
\hline
\end{array}
\tag{3.42}
$$

In the situations where there are more than one fermionic flavor or current, additional Klein factors are required.

3.2 Bosonization of Higher Order Gradients

It will be necessary to bosonize higher-order gradients of fermionic bilinears such as $\psi^\dagger(\partial)^n\psi$ where ∂ is a 1D derivative. These terms naturally arise whenever a fermionic field does not disperse relativistically. See for example Ref. [Haldane, 1981] for a discussion. There is also a subtlety in terms of ordering. The operator ∂ is often interpreted to act on the right but this will often suggest to Noether currents that are not manifestly conserved. The standard textbook example being the definition of the probability current for the non-relativistic Schrödinger equation is $-\frac{i}{2m}\left[\psi^*(\vec{\nabla}\psi) - (\vec{\nabla}\psi^*)\psi\right]$ and not $-\frac{i}{m}\psi^*(\vec{\nabla}\psi)$. Of course they differ only by a boundary term and are equivalent in variational formulation (Rayleigh-Ritz) of the Schrödinger equation. We will adopt the approach of always expressing gradients by the

[2]Notation for the 1+1 chiral boson folows from Chapter 5 of Ref. [Fradkin, 2013]

symmetrized gradient operator

$$\overleftrightarrow{\nabla} := \frac{\vec{\nabla} - \overleftarrow{\nabla}}{2}, \quad \Rightarrow \quad \phi \overleftrightarrow{\nabla} \psi = \frac{1}{2}\Big(\phi(\vec{\nabla}\psi) - (\vec{\nabla}\phi)\psi\Big) \tag{3.43}$$

where the arrows sometimes denote the direction in which the operators act. Also we will neglect all boundary terms in the final identification between fermionic and bosonic operators. The is benign since these gradient terms always appear as integrands inside a full spatial integral.

For fermionic bilinears, a starting point is to consider the short-distance expansion of

$$\psi_S^\dagger(x_1\vec{n}_S + x_t\vec{t}_S)\psi_S(x_2\vec{n}_S + x_t\vec{t}_S) = \frac{i\hbar v_F N(0)}{(x_1 - x_2)} : \mathrm{e}^{i[\phi_S(x_1\vec{n}_S + x_t\vec{t}_S) - \phi_S(x_2\vec{n}_S + x_t\vec{t}_S)]} : \tag{3.44}$$

By translational symmetry we can take the origin $\mathbf{x} = 0$ to be point of interest and $\mathbf{x}_1, \mathbf{x}_2$ to lie close to it. That is we seek to bosonize the equal time operator $\psi_S^\dagger(0)(\overleftrightarrow{\nabla}_S)^n\psi_S(0)$ by considering two points $\mathbf{x}_1, \mathbf{x}_2$ near 0. We next Taylor expand about 0 and then expand the exponential.

$$\begin{aligned}
\psi_S^\dagger(x_1\vec{n}_S)\psi_S(x_2\vec{n}_S) &= \frac{i\hbar v_F N(0)}{(x_1 - x_2)} : \exp\left[\sum_{m=1}^{\infty} \frac{i}{m!}(x_1^m - x_2^m)\phi_S^{(m)}(0)\right] : \\
&= \frac{i\hbar v_F N(0)}{(x_1 - x_2)} : \left\{1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left[\sum_{m=1}^{\infty} \frac{i}{m!}(x_1^m - x_2^m)\phi_S^{(m)}\right]^n\right\} :
\end{aligned} \tag{3.45}$$

Now note that the lowest order term is a c-number and is the term that is subtracted by under the point-split regularization.

$$\langle \psi_S^\dagger(x_1\vec{n}_S)\psi_S(x_2\vec{n}_S)\rangle = \frac{i\hbar v_F N(0)}{(x_1 - x_2)} \tag{3.46}$$

Hence

$$
\begin{aligned}
: \psi_S^\dagger(x_1\vec{n}_S)\psi_S(x_2\vec{n}_S) : &= \psi_S^\dagger(x_1\vec{n}_S)\psi_S(x_2\vec{n}_S) - \langle\psi_S^\dagger(x_1\vec{n}_S)\psi_S(x_2\vec{n}_S)\rangle \\
&= \frac{i\hbar v_F N(0)}{(x_1 - x_2)} : \left\{ \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{m=1}^{\infty} \frac{i}{m!} (x_1^m - x_2^m) \phi_S^{(m)}(0) \right]^n \right\} :
\end{aligned} \tag{3.47}
$$

By a brute force expansion the first few low order terms can be extracted as an expansion by order in m and n. To second order for example we have

$$
\begin{aligned}
: \psi_S^\dagger(x_1\vec{n}_S)\psi_S(x_2\vec{n}_S) : = \frac{i\hbar v_F N(0)}{(x_1 - x_2)} : \Big\{ & i(x_1 - x_2)\phi_S' - \frac{1}{2}(x_1 - x_2)^2(\phi_S')^2 + \frac{i}{2}x_1x_2(x_1 - x_2)(\phi_S')^3 \\
& + \frac{1}{4}x_1^2x_2^2(\phi_S')^4 + \frac{i}{2}(x_1^2 - x_2^2)\phi_S'' + \frac{1}{2}x_1x_2(x_1 + x_2)(\phi_S')(\phi_S'') + \dots \Big\} :
\end{aligned} \tag{3.48}
$$

where $\phi_S' \equiv \nabla_S\phi_S(0)$ etc. Also ambiguously ordered operator expressions such as $(\phi_S')(\phi_S'')$ really correspond to their symmetrized form such as $1/2[(\phi_S')(\phi_S'') + (\phi_S'')(\phi_S)']$. From this expansion, derivatives are taken and leading to the desired bosonized expressions. For example, by denoting $(\#)$ as the RHS of Eq. (3.48) we have

$$
\begin{aligned}
: \psi_S^\dagger\psi_S : &= (\#)|_{x_1=x_2=0} \\
&= -\hbar v_F N(0)(\nabla_S\phi_S)
\end{aligned} \tag{3.49}
$$

$$
\begin{aligned}
: \psi_S^\dagger[\nabla_S\psi_S] : &= \left.\frac{\partial(\#)}{\partial x_2}\right|_{x_1=x_2=0} \\
&= i\hbar v_F N(0) \left[\frac{1}{2}(\nabla_S\phi)^2 + \frac{i}{2}(\nabla_S^2\phi_S) \right].
\end{aligned} \tag{3.50}
$$

The final term of the final equation above is recognized to be a boundary term and is customarily omitted in the bosonization dictionary.

The first non-trivial higher order gradient which requires the next order in expansion of

Eq. (3.48) comes from

$$
\begin{aligned}
\psi_S^\dagger(\overleftrightarrow{\nabla}_S)^2\psi_S &= \psi_S^\dagger\left(\tfrac{1}{4}\right)(\vec{\nabla}_S - \overleftarrow{\nabla}_S)(\vec{\nabla}_S - \overleftarrow{\nabla}_S)\psi_S \\
&= \tfrac{1}{4}\psi_S^\dagger(\vec{\nabla}_S^2 + \vec{\nabla}_S^2 - 2\overleftarrow{\nabla}_S\vec{\nabla}_S)
\end{aligned}
\tag{3.51}
$$

To which we derive

$$
\begin{aligned}
:\psi_S^\dagger(\vec{\nabla}_S^2)\psi_S: &= i\hbar v_F N(0)\left[-\frac{i}{3}(\nabla_S\phi_S)^3 + (\nabla_S\phi_S)(\nabla_S^2\phi_S) + \frac{i}{3}(\nabla_S^3\phi_S)\right] \\
:\psi_S^\dagger(\overleftarrow{\nabla}_S^2)\psi_S: &= i\hbar v_F N(0)\left[-\frac{i}{3}(\nabla_S\phi_S)^3 - (\nabla_S\phi_S)(\nabla_S^2\phi_S) + \frac{i}{3}(\nabla_S^3\phi_S)\right] \\
:\psi_S^\dagger(\overleftarrow{\nabla}_S\vec{\nabla}_S)\psi_S: &= i\hbar v_F N(0)\left[\frac{i}{3}(\nabla_S\phi_S)^3\right]
\end{aligned}
\tag{3.52}
$$

which finally yields

$$
:\psi_S^\dagger(\overleftrightarrow{\nabla}_S)^2\psi_S: = \hbar v_F N(0)\left[\frac{1}{3}(\nabla_S\phi_S)^3 - \frac{1}{6}(\nabla_S^3\phi_S)\right]. \tag{3.53}
$$

Noticing that the last term is again a boundary term which we omit, we then have the following identification

$$
\psi_S^\dagger(\overleftrightarrow{\nabla})^2\psi_S \quad \rightsquigarrow \quad \frac{\hbar v_F N(0)}{3}(\nabla_S\phi_S)^3 \tag{3.54}
$$

The cubic derivative $\psi_S^\dagger(\overleftrightarrow{\nabla}_S)^3\psi_S$ requires a even higher order expansion but which can be facilitated efficiently bu the use of symbolic software such as Mathematica. First

$$
(\overleftrightarrow{\nabla}_S)^3 = \frac{1}{8}(\vec{\nabla}_S^3 - \overleftarrow{\nabla}_S^3 + 3\overleftarrow{\nabla}_S^2\vec{\nabla}_S - 3\overleftarrow{\nabla}_S\vec{\nabla}_S^2) \tag{3.55}
$$

and leads to

$$: \psi_S^\dagger (\vec{\nabla}_S^3) \psi_S : \; = \hbar v_F N(0) \Big[-\frac{i}{4}(\nabla_S \phi_S)^4 + \frac{3}{2}(\nabla_S \phi_S)^2 (\nabla_S^2 \phi_S) + \frac{3i}{4}(\nabla_S^2 \phi_S)^2 + i(\nabla_S \phi_S)(\nabla_S^3 \phi_S) - \frac{1}{4}(\nabla_S^4 \phi_S) \Big] \tag{3.56}$$

$$: \psi_S^\dagger (\overleftarrow{\nabla}_S^3) \psi_S : \; = \hbar v_F N(0) \Big[+\frac{i}{4}(\nabla_S \phi_S)^4 + \frac{3}{2}(\nabla_S \phi_S)^2 (\nabla_S^2 \phi_S) - \frac{3i}{4}(\nabla_S^2 \phi_S)^2 - i(\nabla_S \phi_S)(\nabla_S^3 \phi_S) - \frac{1}{4}(\nabla_S^4 \phi_S) \Big] \tag{3.57}$$

$$: \psi_S^\dagger (\overleftarrow{\nabla}_S^2 \vec{\nabla}_S) \psi_S : \; = \hbar v_F N(0) \Big[-\frac{i}{4}(\nabla_S \phi_S)^4 - \frac{1}{2}(\nabla_S \phi_S)^2 (\nabla_S^2 \phi_S) - \frac{i}{4}(\nabla_S^2 \phi_S)^2 - \frac{1}{12}(\nabla_S^4 \phi_S) \Big] \tag{3.58}$$

$$: \psi_S^\dagger (\overleftarrow{\nabla}_S \vec{\nabla}_S^2) \psi_S : \; = \hbar v_F N(0) \Big[+\frac{i}{4}(\nabla_S \phi_S)^4 - \frac{1}{2}(\nabla_S \phi_S)^2 (\nabla_S^2 \phi_S) + \frac{i}{4}(\nabla_S^2 \phi_S)^2 - \frac{1}{12}(\nabla_S^4 \phi_S) \Big] \tag{3.59}$$

Finally collecting terms and dropping the boundary terms we arrive at the identification

$$\psi_S^\dagger (\overleftrightarrow{\nabla}_S)^3 \psi_S \quad \rightsquigarrow \quad \frac{i\hbar v_F N(0)}{4} (\nabla_S \phi_S)^4 \tag{3.60}$$

3.3 2D Bosonization Dictionary: Quick Lookup

All operators are understood to lie inside $\langle \star \rangle$ as part of a correlator and appropriately normal ordered. The boson here $\phi_{nS}(\vec{x})$ is taken to be a chiral right mover in the $\vec{n}_S$ direction. Also the following shorthand is used

$$\nabla_S \equiv \vec{n}_S \cdot \vec{\nabla},$$
$$\tfrac{\Lambda}{(2\pi)^2} \equiv \hbar v_F N(0),$$
$$x_t \equiv \vec{t}_S \cdot \vec{x},$$
$$x_n \equiv \vec{n}_S \cdot \vec{x}.$$

The operator identifications are as follows:

- $\psi_{nS}(\vec{x}) \quad \rightsquigarrow \quad \eta_{nS}(x_t)\sqrt{\hbar N(0) v_F \lambda} : \mathrm{e}^{-i\phi_S(\vec{x})} :$
- $\psi^\dagger_{nS}(\vec{x})\psi_{nS}(\vec{x}) \quad \rightsquigarrow \quad -\hbar v_F N(0) \nabla_S \phi_{nS}(\vec{x})$
- $\psi^\dagger_{nS}(\vec{x})\psi_{mS}(\vec{x}) \quad \rightsquigarrow \quad (\hbar v_F N(0)\lambda)\, \eta_{nS}(x_t)\eta_{mS}(x_t) : \mathrm{e}^{i[\phi_{1S}(\vec{x})-\phi_{2S}(\vec{x})]} : \quad m \neq n$
- $\psi^\dagger_{nS}(\vec{x})\left(-i\overleftrightarrow{\nabla}_S \psi_{nS}(\vec{x})\right) \quad \rightsquigarrow \quad \left[\frac{\hbar v_F N(0)}{2}\right] (\nabla_S \phi_{nS}(\vec{x}))^2$
- $\psi^\dagger_{nS}(\vec{x})\left(-\overleftrightarrow{\nabla}^2_S \psi_{nS}(\vec{x})\right) \quad \rightsquigarrow \quad -\frac{\hbar v_F N(0)}{3} (\nabla_S \phi_{nS}(\mathbf{x}))^3$
- $\psi^\dagger_{nS}(\vec{x})\left(i\overleftrightarrow{\nabla}^3_S \psi_{nS}(\vec{x})\right) \quad \rightsquigarrow \quad \frac{\hbar v_F N(0)}{4} (\nabla_S \phi_{nS}(\mathbf{x}))^4$

The nematic order parameters $\phi_{li\mu}$ defined by

$$\phi_{li\mu} := \int d^2x \sum_S \begin{Bmatrix} \cos(l\theta_S) \\ \sin(l\theta_S) \end{Bmatrix} \sigma^\mu_{mn} \langle : \psi^\dagger_{mS}(\vec{x})\psi_{nS}(\vec{x}) : \rangle, \quad i = 1 \text{ top and } i = 2 \text{ bottom}$$

which are not to be confused with the bosonized fields ϕ_{nS}, acquire the following bosonized forms:

- $\phi_{lix} \quad \rightsquigarrow \quad +2\hbar v_F N(0)\lambda \int d^2x \sum_S \begin{Bmatrix} \cos(l\theta_S) \\ \sin(l\theta_S) \end{Bmatrix} \langle (i\eta_{1S}\eta_{2S}) \sin[\phi_{1S} - \phi_{2S}] \rangle$

- $\phi_{liy} \quad \rightsquigarrow \quad -2\hbar v_F N(0)\lambda \int d^2x \sum_S \begin{Bmatrix} \cos(l\theta_S) \\ \sin(l\theta_S) \end{Bmatrix} \langle (i\eta_{1S}\eta_{2S}) \cos[\phi_{1S} - \phi_{2S}] \rangle$

Succinctly one can define a vector valued field

$$\vec{\varphi}_{12S} := (\cos[\phi_{1S} - \phi_{2S}], \sin[\phi_{1S} - \phi_{2S}], 0)^T$$

which leads to the bosonization translation

- $\phi_{li\mu} \quad \rightsquigarrow \quad 2\hbar v_F N(0)\lambda \int d^2x \sum_S \begin{Bmatrix} \cos(l\theta_S) \\ \sin(l\theta_S) \end{Bmatrix} \epsilon_{\mu\nu} \langle (i\eta_{1S}\eta_{2S})(\vec{\varphi}_{12S})_\nu \rangle$

The operator $(\vec{\varphi}_{12S})_x + i(\vec{\varphi}_{12S})_y = \mathrm{e}^{i[\phi_{1S} - \phi_{2S}]}$ is the local phase difference operator between bands for the patch direction S. The mean fields $\vec{\phi}_{l1}, \vec{\phi}_{l2}$ are harmonic resolved versions of this up to the fermion parity operator $i\eta_{1S}\eta_{2S}$ coming from the Klein factors. A useful identity for patch harmonics is

$$\sum_{l=0}^{N-1} \begin{Bmatrix} \cos(l\theta_S)\cos(l\theta_T) \\ \sin(l\theta_S)\sin(l\theta_T) \end{Bmatrix} = \frac{N\delta_{ST}}{2}$$

since $\mathrm{e}^{i\theta_S}$ is an N-th root of 1. Here N is the number of patches.

3.4 Bosonized Model

To re-iterate the fermionic hamiltonian is that of two-dimensional metal of Eq. (2.2)

$$\begin{aligned} H &= H_{\text{kin}} + H_{\text{int}} && (3.61) \\ &= \sum_{\mathbf{k},n} \psi^\dagger_{n\mathbf{k}} (\varepsilon_n(|\mathbf{k}|) - \mu)\psi_{n\mathbf{k}} + \sum_{\mathbf{q}} \left(\frac{f_l(\mathbf{q})}{2} \right) \sum_{i=1,2} \sum_{\mu=x,y} \Phi_{li\mu}(\mathbf{q})\Phi_{li\mu}(-\mathbf{q}) && (3.62) \end{aligned}$$

The Fermi bilinears are

$$\Phi_{l1\mu}(\mathbf{q}) = \sum_{\mathbf{k},m,n} \cos(l\theta_{\mathbf{k}}) : \psi^{\dagger}_{m\mathbf{k}+\mathbf{q}/2}\sigma^{\mu}_{mn}\psi_{n\mathbf{k}-\mathbf{q}/2} : \tag{3.63}$$

$$\Phi_{l2\mu}(\mathbf{q}) = \sum_{\mathbf{k},m,n} \sin(l\theta_{\mathbf{k}}) : \psi^{\dagger}_{m\mathbf{k}+\mathbf{q}/2}\sigma^{\mu}_{mn}\psi_{n\mathbf{k}-\mathbf{q}/2} : \tag{3.64}$$

where $\sigma^{x,y}$ are two of the three Pauli matrices.

Although we intend to study the low energy properties of this model – specifically the physics near the chemical potential μ – we require non-linearities in the dispersion $\epsilon_{1,2}(|\mathbf{k}|)$. To this end we expand the dispersion relation near k_F to third order in deviation of momentum as follows

$$\epsilon_{1,2}(k) - \mu = \epsilon(k) - \mu \pm \hbar v_F \Delta \tag{3.65}$$

with

$$\begin{aligned}\epsilon(k) - \mu &= \epsilon'(k_F)\delta k + \frac{1}{2!}\epsilon''(k_F)\delta k^2 \quad + \frac{1}{3!}\epsilon'''(k_F)\delta k^3 \\ &= \quad \hbar v_F \delta k \quad + \frac{1}{2}(\hbar v_F)^2 c_1 \delta k^2 + \frac{1}{3!}(\hbar v_F)^3 c_2 \delta k^3\end{aligned} \tag{3.66}$$

where $\delta k := k - k_F$. The “Fermi pressure" terms $\pm\hbar v_F\Delta$ act to move the Fermi momentum of bands 1 and 2 from k_F as shown in Fig. 2.1 when the higher order gradient corrections are exactly zero $c_1 = c_2 = 0$. The dispersion parameters v_F, c_1 and c_2 are related to the gradients in dispersion and derivatives of the density of states $N(0)$ as follows

$$\epsilon'(k_F) = \hbar v_F \tag{3.67}$$

$$\epsilon''(k_F) = (\hbar v_F)^2 c_1 = (\hbar v_F)^2 \left(-\frac{N'(0)}{N(0)}\right) \tag{3.68}$$

$$\epsilon'''(k_F) = (\hbar v_F)^3 c_2 = (\hbar v_F)^3 \left[3\left(\frac{N'(0)}{N(0)}\right)^2 - \frac{N''(0)}{N(0)}\right]. \tag{3.69}$$

These expressions are inferred from the following relation previously derived

$$\epsilon'(k_F) = \hbar v_F = \left[\frac{\Lambda}{(2\pi)^2}\right]\frac{1}{N(0)} \tag{3.70}$$

where $N(E)$ is density of states[3] as a function of energy. This then leads to the following expressions for the kinetic Hamiltonian, which includes a coarse-grained version in terms of patched fermions

$$\begin{aligned} H_{\text{kin}} =& \sum_{\mathbf{k},m,n} \psi^\dagger_{n\mathbf{k}} \Bigg[\hbar v_F(k-k_F)\delta_{mn} + \hbar v_F \Delta\sigma^z_{mn} + \left(\frac{\hbar^2 v_F^2 c_1}{2}\right)(k-k_F)^2\delta_{mn} \\ &+ \left(\frac{\hbar^3 v_F^3 c_2}{6}\right)(k-k_F)^3\delta_{mn}\Bigg]\psi_{n\mathbf{k}} \\ =& \sum_S \int d^2x \left(\psi^\dagger_{1S}(\mathbf{x}), \psi^\dagger_{2S}(\mathbf{x})\right)\Bigg\{-i\hbar v_F\nabla_S + \hbar v_F\Delta\sigma^z - \left(\frac{\hbar^2 v_F^2 c_1}{2}\right)\nabla_S^2 \\ &+ \left(\frac{i\hbar^3 v_F^3 c_2}{6}\right)\nabla_S^3\Bigg\}\begin{pmatrix}\psi_{1S}(\mathbf{x}) \\ \psi_{2S}(\mathbf{x})\end{pmatrix} \end{aligned} \tag{3.71}$$

where $\nabla_S \equiv \vec{n}_S \cdot \vec{\nabla}$ is understood to also mean $\overleftrightarrow{\nabla}_S$.

Moving on, H_{int} needs to be translated into coarse-grained patched fermions. Explicitly the order parameters gain the following form

$$\tilde{\Phi}_{l1\mu}(\mathbf{x}) = \sum_S \cos(l\theta_S) : \psi^\dagger_{mS}(\mathbf{x})\sigma^\mu_{mn}\psi_{nS}(\mathbf{x}) : \tag{3.72}$$

$$\tilde{\Phi}_{l2\mu}(\mathbf{x}) = \sum_S \sin(l\theta_S) : \psi^\dagger_{mS}(\mathbf{x})\sigma^\mu_{mn}\psi_{nS}(\mathbf{x}) : \tag{3.73}$$

[3]A small technicality is that the true total density of state is $N \times N(E)$ where N is number of patches which should scale as k_F. So despite $N(E)$ having the units of a 2D density of states (energy $\times$ area)$^{-1}$, it really behaves more like a 1D density of states, that which is associated to the patch fermion ψ_S.

which produces

$$
\begin{aligned}
H_{\text{int}} &= \sum_{\mu=x,y} \int d^2x \int d^2x' \, \frac{\tilde{f}(\mathbf{x}'-\mathbf{x})}{2} \tilde{\Phi}_{li\mu}(\mathbf{x}')\tilde{\Phi}_{li\mu}(\mathbf{x}) \\
&= \sum_{\mu=x,y} \int d^2x' \int d^2x \frac{\tilde{f}(\mathbf{x}'-\mathbf{x})}{2} \sum_{S,T} \Big\{ \cos(l\theta_S)\cos(l\theta_T) :\psi_S^\dagger(\mathbf{x}')\sigma^\mu\psi_S(\mathbf{x}')::\psi_T^\dagger(\mathbf{x})\sigma^\mu\psi_T(\mathbf{x}): \\
&\qquad\qquad + \sin(l\theta_S)\sin(l\theta_T) :\psi_S^\dagger(\mathbf{x}')\sigma^\mu\psi_S(\mathbf{x}')::\psi_T^\dagger(\mathbf{x})\sigma^\mu\psi_T(\mathbf{x}): \Big\} \\
&= \sum_{\mu=x,y} \int d^2x' \int d^2x \frac{\tilde{f}(\mathbf{x}'-\mathbf{x})}{2} \sum_{S,T} \cos[l(\theta_S-\theta_T)] :\psi_S^\dagger(\mathbf{x}')\sigma^\mu\psi_S(\mathbf{x}')::\psi_T^\dagger(\mathbf{x})\sigma^\mu\psi_T(\mathbf{x}):
\end{aligned}
\tag{3.74}
$$

3.4.1 Bosonized Hamiltonian

Next applying the bosonization dictionary yields the following bosonized kinetic Hamiltonian

$$
\begin{aligned}
H_{\text{kin}} = \hbar v_F N(0) \sum_S \int d^2x \Bigg\{ \left(\frac{\hbar v_F}{2}\right) \left[(\nabla_S\phi_{1S})^2 + (\nabla_S\phi_{2S})^2\right] - \hbar v_F \Delta \left[\nabla_S\phi_{1S} - \nabla_S\phi_{2S}\right] \\
- \left(\frac{\hbar^2 v_F^2 c_1}{3!}\right) \left[(\nabla_S\phi_{1S})^3 + (\nabla_S\phi_{2S})^3\right] + \left(\frac{\hbar^3 v_F^3 c_2}{4!}\right) \left[(\nabla_S\phi_{1S})^4 + (\nabla_S\phi_{2S})^4\right] \Bigg\}
\end{aligned}
\tag{3.75}
$$

It is important to note that the bosonic fields here ϕ_{1S}, ϕ_{2S} are defined relative to a common Fermi surface of radius k_F as shown in Fig. 2.1, and not of the shifted Fermi surfaces. The interaction part of the Hamiltonian can be decoupled using the Hubbard-Stratonovich transformation. Introducing the Hubbard-Stratonovich fields, $\Gamma_{li\mu}(x)$ yields

$$
\begin{aligned}
H_{int}(\Gamma_{li\mu}, \tilde{\Phi}_{li\mu}) &= \int d^2x \{\Gamma_{l1x}(x)\tilde{\Phi}_{l1x}(x) + \Gamma_{l1y}(x)\tilde{\Phi}_{l1y}(x) + \Gamma_{l2x}(x)\tilde{\Phi}_{l2x}(x) + \Gamma_{l2y}(x)\tilde{\Phi}_{l2y}(x)\} \\
&\quad - \int d^2x \int d^2x' \frac{f_l^{-1}(x'-x)}{2} \sum_{i\mu} \Gamma_{li\mu}(x)\Gamma_{li\mu}(x')
\end{aligned}
\tag{3.76}
$$

We can then integrate out the Hubbard-Stratonovich fields, Γ, to obtain the Hamiltonian, which depends only on the bosonic degrees of freedom. One has to consider two possibilities: the inter-patch and the intra-patch interaction cases. For the inter-patch case, the Hamiltonian contains the scattering between two different patches, which in the bosonized language, has a form of coupled sine-Gordon terms.

Finally the bosonization dictionary applied to the interaction term H_{int} requires some care due to an intra-patch term, and the details of its derivation are relegated to Appendix A.1. The final bosonized form is as follows

$$\boxed{\begin{aligned} H_{\text{int}} = 2[\hbar v_F N(0)\lambda]^2 \int d^2x' \int d^2x \tilde{f}_l(\mathbf{x}' - \mathbf{x}) \sum_{S,T} \cos[l(\theta_S - \theta_T)] : \cos[\Delta\phi_S(\mathbf{x}') - \Delta\phi_T(\mathbf{x})] : \\ + [\hbar v_F N(0)]^2 f_l(0) L^2 \int d^2x \sum_S [\nabla_S \phi_{1S}(\mathbf{x}) - \nabla_S \phi_{2S}(\mathbf{x})]^2 \end{aligned}} \tag{3.77}$$

where $\Delta\phi_S(\mathbf{x}) := \phi_{1S}(\mathbf{x}) - \phi_{2S}(\mathbf{x})$ and is not to be confused with the splitting momentum Δ. We should also remark that this bosonized form was derived by setting the unitary operators formed($\eta_m \eta_n$) from Klein operators to unity. It is important to identify the interaction Hamiltonian as having two different terms: an inter-pathch and an intra-path contributions. The inter-patch contribution dominates the phase winding and drives the β_1 phase while the intra-patch term not only introduces a correction to the kinetic energy but also a density-density scattering piece that drives the phase distortion, which is a characteristic of the α_1 phase.

3.5 Results from Bosonization

This section summarizes the results of the bosonization approach. It focuses on the correction of the Pomerunchuk critical point which comes in this approace by analyzing the quadratic part of the action. The β_1 phase ground state is also derived.

3.5.1 Pomerunchuk Instability

We can proceed in two ways to analyze the phases described by Eq. (3.75) and Eq. (3.77). The first and most natural route would be appeal to an action-functional integral reformulation of the bosonized $H = H_{\text{kin}} + H_{\text{int}}$. However care must be taken in this case since the bosonic chiral fields $\phi_{mS}(\mathbf{x})$ are non-local since its commutator between two space-time points will not commute[4] in general. Moreover as we shall discuss, $\phi_{mS}(\mathbf{x})$ should always contain a term which is linear in $x_n = \vec{n}_S \cdot \mathbf{x}$ and is proportional to the density operator. However for purely technical reasons, we will opt for the second approach which is to determine optimal variational ground state wavefunctions.

From the bosonization dictionary we have the following

$$\delta n_{mS}(\mathbf{x}) = :\psi^\dagger_{mS}(\mathbf{x})\psi_{mS}(\mathbf{x}) := -\hbar v_F N(0)\nabla_S \phi_{mS}(\mathbf{x}). \tag{3.78}$$

When $\delta n_{mS}(\mathbf{x}) = \delta n_{mS}$ is static and uniform, that is at a fixed filling of ψ_{mS}, we may identify the c-number δn_{mS} with a gradient term in $\phi_{mS}(\mathbf{x})$. From the proportionality constants required of the bosonization correspondence, we write

$$\phi_{mS}(\mathbf{x}) = \tilde{\phi}_{mS}(\mathbf{x}) + \phi^{(0)}_{mS} - \delta k_{mS}\,(\vec{n}_S \cdot \mathbf{x}) \tag{3.79}$$

where

$$\delta k_{mS} := \frac{\delta n_{mS}}{\hbar v_F N(0)} \tag{3.80}$$

is to be interpreted as a shift in the Fermi momentum k_F such that $\psi_{mS} \propto \mathrm{e}^{i\delta k_{mS} x_n} : \mathrm{e}^{-i[\tilde{\phi}_{mS} - \phi^{(0)}_{mS}]} :$. The operators δn_{mS} and δk_{mS} are analogues of the number operator $\hat{N}$ in 1D bosonization which makes an appearance in the operator identities in a finite system

[4]Even in the simplest 1D bosonization case where ϕ_S is akin to a right-mover, one has at equal time $[\phi_R(x), \phi_R(y)] = \frac{i}{4}\text{sign}(x-y)$ for right movers.

of size L. The c-number $\phi_{mS}^{(0)}$is the constant phase of ψ_{mS}, which is required to make U(1) relative phase symmetry manifest[Sun and Fradkin, 2008]. The fields or operators $\tilde{\phi}_{mS}(\mathbf{x})$ are harmonic (periodic) in $\mathbf{x}$-space and represent particle-hole excitations.

Next we select the following form of variational ansatz for the bosonic ground state[5]

$$|\Psi\rangle := |\{\phi_{1S}^{(0)}, \delta n_{1S}\}, \{\phi_{2S}^{(0)}, \delta n_{2S}\}\rangle \tag{3.81}$$

with the properties that for all S and $\mathbf{x}$

$$\begin{aligned} i\eta_{1S}(x_t)\eta_{2S}(x_t)|\Psi\rangle &= |\Psi\rangle \\ \widehat{\delta n}_{mS}|\Psi\rangle &= \delta n_{mS}|\Psi\rangle \\ \widehat{\phi_{mS}^{(0)}}|\Psi\rangle &= \phi_{mS}^{(0)}|\Psi\rangle \\ \widehat{\tilde{\phi}}_{mS}^{+}(\mathbf{x})|\Psi\rangle &= 0. \end{aligned} \tag{3.82}$$

where the hats are inserted to emphasize operators on the left hand sides. The last condition encodes the physical statement that there are no particle-hole excitations and translational symmetry remains unbroken. This is an assumption based on the expectation that it costs energy to create particle-hole pairs but is otherwise unjustified. The first condition should also be regarded as non-trivial since it sets a parity of the wavefunction under exchange of ψ_{1S} and ψ_{2S} which related to the Majoranic interpretation of the Klein factors. With these caveats, our task next is to determine the optimal variational form of $|\Psi\rangle$.

3.5.2 Variational Energies

Direct substitution of the form of the ansatz(Eq. (3.81)) and its properties(Eq. (3.82)) into $E_0 := \langle\Psi|H_{\text{kin}}+H_{\text{int}}|\Psi\rangle$ produces the following variational forms of the kinetic and interaction energies

[5] Note this is not the ground state to which the normal ordering $:*:$ is defined by.

$$\langle\Psi|H_{\text{kin}}|\Psi\rangle = L^2 \sum_S \left\{ \left(\frac{c_2}{4!N(0)^3}\right) \left[(\delta n_{1S})^4 + (\delta n_{2S})^4\right] + \left(\frac{c_1}{3!N(0)^2}\right) \left[(\delta n_{1S})^3 + (\delta n_{2S})^3\right] \right.$$
$$\left. + \left(\frac{1}{2N(0)}\right) \left[(\delta n_{1S})^2 + (\delta n_{2S})^2\right] + (\hbar v_F \Delta) \left[\delta n_{1S} - \delta n_{2S}\right] \right\} \tag{3.83}$$

$$\langle\Psi|H_{\text{int}}|\Psi\rangle = L^2 \sum_S f_l(0) L^2 \left[\delta n_{1S} - \delta n_{2S}\right]^2$$
$$+ 2[\hbar v_F N(0)]^2 \lambda^2 \int d^2x' \int d^2x \tilde{f}_l(\mathbf{x}' - \mathbf{x}) \sum_{S,T}$$
$$\cos[l(\theta_S - \theta_T)] \cos\left[\frac{(\delta n_{1S} - \delta n_{2S}) x'_{Sn} - (\delta n_{1T} - \delta n_{2T}) x_{Tn}}{\hbar v_F N(0)} - \Delta\phi_S^{(0)} + \Delta\phi_T^{(0)}\right] \tag{3.84}$$

where $x'_{SN} := \vec{n}_S \cdot \mathbf{x}'$ and $x_{Tn} := \vec{n}_T \cdot \mathbf{x}$. If expressed in terms of δk_{mS} and combining both contributions, we arrive at the following neater expression of the variational energy density

$$\varepsilon_0(\{\phi_{mS}^{(0)}, \delta k_{mS}\}) := \frac{\langle\Psi|H|\Psi\rangle}{(\hbar v_F N(0) L)^2}$$
$$= \varepsilon_{\text{kin}} + \varepsilon_{\text{int}} \tag{3.85}$$

where

$$\varepsilon_{\text{kin}} = \sum_S \left\{ \left(\frac{\Delta}{N(0)}\right) \left[\delta k_{1S} - \delta k_{2S}\right] + f_l(0) L^2 \left[\delta k_{1S} - \delta k_{2S}\right]^2 + \frac{1}{2N(0)} \left[(\delta k_{1S})^2 + (\delta k_{2S})^2\right] \right.$$
$$\left. + \left(\frac{\hbar v_F c_1}{3!N(0)}\right) \left[(\delta k_{1S})^3 + (\delta k_{2S})^3\right] + \left(\frac{\hbar^2 v_F^2 c_2}{4!N(0)}\right) \left[(\delta k_{1S})^4 + (\delta k_{2S})^4\right] \right\} \tag{3.86}$$

and

$$\varepsilon_{\text{int}} = 2\lambda^2 \int dx'_n \int dx_n \tilde{f}_l(\mathbf{x}' - \mathbf{x}) \sum_{S,T} \cos[l(\theta_S - \theta_T)] \cos\left[\delta k_{12S} x'_n - \delta k_{12T} x_n - \Delta\phi_S^{(0)} + \Delta\phi_T^{(0)}\right] \tag{3.87}$$

where $\delta k_{12S} := \delta k_{1S} - \delta k_{2S}$ and $\Delta\phi_S^{(0)} := \phi_{1S}^{(0)} - \phi_{2S}^{(0)}$. Also in the final line of the interaction contribution, the integral over transverse directions x'_t, x_t has been taken since the integrand is independent of these coordinates.

3.5.3 Phases and Their Stability

To examine the phases described within this space of variational ground states, we need an analysis of the extremal (saddle point) solutions to the variational energy (Eq. (3.83) and Eq. (3.84)).

The Normal Phase

The normal phase is a phase where the nematic order has not been established. That is, there is neither broken isotropy symmetry in δn_{mS} nor $\phi_{mS}^{(0)}$. This state of affairs should be captured by the following conditions

$$\delta n_{1S} = \delta n_1, \quad \delta n_{2S} = \delta n_2 \tag{3.88}$$

$$\phi_{1S}^{(0)} = \phi_1^{(0)}, \quad \phi_{2S}^{(0)} = \phi_2^{(0)} \tag{3.89}$$

for all S. Thus the inter-patch interaction contribution simplifies to

$$2[\hbar v_F N(0)]^2 f_l(0) L^2 \lambda^2 \int dx'_n \int dx_n \sum_{S,T} \underbrace{\cos[l(\theta_S - \theta_T)]}_{=0} \cos\left[\frac{(\delta n_1 - \delta n_2)(x'_n - x_n)}{\hbar v_F N(0)}\right] = 0 \tag{3.90}$$

where the cancellation from the $l \neq 0$ partial wave harmonics leads to the zero result. Hence the following saddle points equations are produced

$$\left(\frac{c_2}{6N(0)}\right)(\delta n_1)^3 + \left(\frac{c_1}{2N(0)}\right)(\delta n_1)^2 \left(\frac{1}{N(0)} + 2f_l(0)b^2\right)\delta n_1 - [2f_l(0)b^2]\delta n_2 + \hbar v_F \Delta = 0 \tag{3.91}$$

$$\left(\frac{c_2}{6N(0)}\right)(\delta n_2)^3 + \left(\frac{c_1}{2N(0)}\right)(\delta n_2)^2 \left(\frac{1}{N(0)} + 2f_l(0)b^2\right)\delta n_2 - [2f_l(0)b^2]\delta n_1 - \hbar v_F \Delta = 0 \tag{3.92}$$

Defining a "stiffness matrix"

$$\mathcal{A} = \begin{pmatrix} 1 + 2f_l(0)b^2N(0) & -2f_l(0)b^2N(0) \\ -2f_l(0)b^2N(0) & 1 + 2f_l(0)b^2N(0) \end{pmatrix} \tag{3.93}$$

we can rearrange the saddle point equations into a self-consistent form

$$\begin{pmatrix} \delta n_1 \\ \delta n_2 \end{pmatrix} = [\hbar v_F N(0)]\mathcal{A}^{-1}\begin{pmatrix} \Delta \\ -\Delta \end{pmatrix} - \left(\frac{c_1}{2N(0)}\right)\mathcal{A}^{-1}\begin{pmatrix} \delta n_1^2 \\ \delta n_2^2 \end{pmatrix} - \left(\frac{c_2}{6N(0)^2}\right)\mathcal{A}^{-1}\begin{pmatrix} \delta n_1^3 \\ \delta n_2^3 \end{pmatrix}. \tag{3.94}$$

We imagine working in the regime where $\hbar v_F \Delta c_1, \hbar^2 v_F^2 \Delta^2 c_2 \ll 1$ but with $c_2 > 0$ for stability reasons. The coupled cubic saddle point equations are not solvable analytically for general parameters. Special limits like $c_1 = \Delta = 0$ and $\delta n_1 = \delta n_2$ have closed form solutions though. In the simplest scenario when the higher order dispersion effects are neglected $c_1 = c_2$ the solution to Eq. (3.94) is clearly $\delta n_1 = -[\hbar v_F N(0)]\Delta$ and $\delta n_2 = +[\hbar v_F N(0)]\Delta$. If we replace the RHS of Eq. (3.94) with this solution, we arrive at the lowest order correction to the normal density occupation distribution

$$
\begin{aligned}
\delta n_1 &\approx -[\hbar v_F N(0)]\Delta - \left(\frac{\hbar^2 v_F^2 N(0) c_1}{2[1+4f_l(0)b^2 N(0)]}\right)\Delta^2 + \left(\frac{\hbar^3 v_F^3 N(0) c_2}{6}\right)\Delta^3 \\
\delta n_2 &\approx +[\hbar v_F N(0)]\Delta - \left(\frac{\hbar^2 v_F^2 N(0) c_1}{2[1+4f_l(0)b^2 N(0)]}\right)\Delta^2 - \left(\frac{\hbar^3 v_F^3 N(0) c_2}{6}\right)\Delta^3
\end{aligned}
\tag{3.95}
$$

Note these expression diverge when $f_l(0) = -[4b^2 N(0)]^{-1}$ unless $c_1 = 0$. In fact computing the Hessian of the variational energy E_0 at this solution, again to lowest oder in c_1 and c_2, yields a matrix with eigenvalues

$$
\frac{1+(\hbar v_F \Delta)^2 c_2}{N(0)}, \quad \frac{1+(\hbar v_F \Delta)^2 c_2 + 4f_l(0)b^2 N(0)}{N(0)} \tag{3.96}
$$

from which we infer the instability of the normal phase to be signaled by the condition that

$$
\boxed{f_l(0) \leq -\left(\frac{1+(\hbar v_F \Delta)^2 c_2}{4b^2 N(0)}\right)} \tag{3.97}
$$

where

$$
c_2 = 3\left(\frac{N'(0)}{N(0)}\right)^2 - \frac{N''(0)}{N(0)}. \tag{3.98}
$$

When the Landau interaction strength $f_l(0)$ is in this regime, we would expect the normal phase to be destabilized in favor of a new extrema away from Eq. (3.95).

3.5.4 β_1 phase

From these fermi bilinear:

$$\Phi_{l,1,\mu}(q) = \sum_{k,n,m} :\psi_n^\dagger(k+q/2)\cos(l\theta_k)\sigma_\mu^{nm}\psi_m(k-q/2): \tag{3.99}$$

$$\Phi_{l,2,\mu}(q) = \sum_{k,n,m} :\psi_n^\dagger(k+q/2)\sin(l\theta_k)\sigma_\mu^{nm}\psi_m(k-q/2): \tag{3.100}$$

The order parameters from the mean-field theory after bosonization are these two component vectors

$$\vec{\Phi}_{l,1} = \sum_S 2i\lambda N(0)\hbar v_F \cos(l\theta_S)\begin{pmatrix} \sin(\phi_{S,2}-\phi_{S,1}) \\ -\cos(\phi_{S,2}-\phi_{S,1}) \end{pmatrix} \tag{3.101}$$

$$\vec{\Phi}_{l,2} = \sum_S 2i\lambda N(0)\hbar v_F \sin(l\theta_S)\begin{pmatrix} \sin(\phi_{S,2}-\phi_{S,1}) \\ -\cos(\phi_{S,2}-\phi_{S,1}) \end{pmatrix} \tag{3.102}$$

To compare the bosonized theory to the mean-field approach, it is natural to see whether the order parameter in bosonized langauge can explain the different phases. First we compute the classical ground state by extremizing the action.

$$\int d^2x'(2\lambda N(0)\hbar v_F)^2\sum_T \cos l(\theta_S-\theta_T)\{\cos\Delta\phi_S(x)\sin\Delta\phi_T(x')-\sin\Delta\phi_S(x)\cos\Delta\phi_T(x')\} = 0 \tag{3.103}$$

This is a nonlinear equation which is difficult to solve. However we can make an ansatz by drawing from the properties we know from the mean-field solutions. We learn that in β phase the order parameters have the same norm and are perpendicular to each other. This amount to two conditions on top of the equation of motion that the solution has to satisfie. From mean-field theory solution[Sun and Fradkin, 2008], the solution is of the uniform undistorted fermi surface but with a non-trivial winding of the relative phases of the fermions in two bands. Since the bosonic fields are the phase of the fermions, we know that part of $\Delta\phi_S$ has to be $l\theta_S$. This gives the winding. However one has to take into account the condensation in the density. Hence another part of $\Delta\phi_S$ is $D_S(x) = (\delta k_{1S} - \delta k_{2S})(\vec{n}_s \cdot \vec{x})$ Therefore the ansatz is

$$\Delta\phi_S = D_S(x) + l\theta_S \tag{3.104}$$

For the sake of algebraic mannipulation we define $\kappa = (\delta k_{1S} - \delta k_{2S})$. Hence

$$D_S(x) = \kappa_x \cos\theta_S + \kappa_y \sin\theta_s$$

We also further define:

$$\alpha = \arctan(\kappa_x/\kappa_y)$$
$$z = \sqrt{\kappa_x^2 + \kappa_y^2}$$

To get a consistent ground state solution it is amount to doing a 3-steps consistency check:

1. Equation of motion.
2. Orthogonality condition, $\vec{\Phi}_{l,1} \cdot \vec{\Phi}_{l,2} = 0$. In fermionic picture, this implies that

$$\sum_{k,k'} [\psi_1^\dagger(k)\psi_2(k)\psi_2^\dagger(k')\psi_1(k') + \psi_2^\dagger(k)\psi_1(k)\psi_1^\dagger(k')\psi_2(k')] \times \cos(l\theta_k)\sin(l\theta_{k'}) = 0 \tag{3.105}$$

We can bosonize this equation to give

$$\begin{aligned}\sum_{S,T}\alpha[e^{i(\Delta\phi_T-\Delta\phi_S)/\hbar}+e^{i(\Delta\phi_T-\Delta\phi_S)/\hbar}]\times\cos(l\theta_S)\sin(l\theta_T) &= 0 \\ \frac{1}{2}\sum_{S,T}\alpha\cos\frac{(\Delta\phi_T-\Delta\phi_S)}{\hbar}\times\sin(l\theta_T+l\theta_S) &= 0 \end{aligned} \tag{3.106}$$

3. Equal norm condition,

   $|\vec{\Phi}_{l,1}| = |\vec{\Phi}_{l,2}|$. This implies that

$$\begin{aligned}&\sum_{S,T}\cos(l\theta_S)\cos(l\theta_T)[\sin\frac{\Delta\phi_S}{\hbar}\sin\frac{\Delta\phi_T}{\hbar}+\cos\frac{\Delta\phi_S}{\hbar}\cos\frac{\Delta\phi_T}{\hbar}] = \\ &\quad\sum_{S,T}\sin(l\theta_S)\sin(l\theta_T)[\sin\frac{\Delta\phi_S}{\hbar}\sin\frac{\Delta\phi_T}{\hbar}+\cos\frac{\Delta\phi_S}{\hbar}\cos\frac{\Delta\phi_T}{\hbar}]\end{aligned} \tag{3.107}$$

We can simplify this to be

$$\sum_{S,T}\cos(l\theta_S)\cos(l\theta_T)\times\cos\frac{(\Delta\phi_S-\Delta\phi_T)}{\hbar} = \sum_{S,T}\sin(l\theta_S)\sin(l\theta_T)\times\cos\frac{(\Delta\phi_S-\Delta\phi_T)}{\hbar} \tag{3.108}$$

For $l = 2$ plugging in the ansatz(Eq.(3.104)) to the definition of the order parameter yields

$$\begin{aligned}\phi_{l1} &= 2i\lambda N(0)\hbar v_F\begin{pmatrix}-J_4(z)\sin 4\alpha \\ -(\pi J_0(z)+J_4\cos 4\alpha)\end{pmatrix} \\ \phi_{l2} &= 2i\lambda N(0)\hbar v_F\begin{pmatrix}\pi J_0(z)-J_4(z)\cos 4\alpha \\ J_4(z)\sin 4\alpha)\end{pmatrix}\end{aligned} \tag{3.109}$$

where $J_n(z)$ is a Bessel function of the first kind. This satisfies the equation of motion(Eq.(3.103)) but does not satisfy the second and third conditions. Therefore it is not a consistent solution.

However if only the condensation in the phase is allowed, $\Delta\phi_S = l\theta_S$, then one can obtain

$$\begin{aligned} \vec{\phi}_1 &= 2i\lambda N(0)\hbar v_F \frac{\pi}{l} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \vec{\phi}_2 &= 2i\lambda N(0)\hbar v_F \frac{\pi}{l} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned} \tag{3.110}$$

This satisfies both the magnitude and the orthogonality conditions(Eq. (3.107) and Eq. (3.106)) on top of the equation of motion. Therefore it is consistent.

Next, one can compute the anomalous Hall conductance for the β_1 phase in the bosonized language. In the mean-field treatment[Sun and Fradkin, 2008], the β_1 demonstrates the unquantized-spontaneous anomalous Hall conductance, while the α_1 phase does not. Furthermore, the Hall conductance is related to the the Berry phase. We would like to start out from our mean-field solution of the β_1 phase and then calculate the equivalent result in the bosonized language. We hope to see two things: first, the conductance is non-zero and un-quantized; second, it depends on the Berry phase. To do this, we study the linear response to the external electric field. This can be calculated by solving the equations of motion with the term coupled to the electromagnetic fields. First, we have to go back to the fermionic Lagrangian and consider minimal coupling to get a guage invariant current. This is done by promoting

$$\partial_\tau \to D_\tau \quad \text{and} \quad \partial_i \to \quad D_i \quad \text{where} \quad D_\tau = \partial_\tau - ie\phi \quad , D_i = \partial_i - ieA_i.$$

In this analysis both the kinetic and the quadrupolar terms have to be promoted. We can bosonize these expressions and add onto our action. From this we can rewrite the saddle-point equations. The sketch of the steps will be the following: 1. Solve these new saddle point equation for the classical fields; 2. Substitute the classical field back in the action to find $Z(\phi_{saddle})$; 3. Use $J_{cl} = \frac{1}{Z_{cl}} \frac{\delta Z_{saddle}}{\delta A}$. We can linearize the equations of motions by expanding

$$\phi_{S,n} = \phi_{S,n}^{MF} + \tilde{\phi}_{S,n}$$

We can solve consistently for the correction. It turns out that the consistent ground state solution that considers only the phase winding(Eq. (3.110)) gives vanishing Hall conductivity.

3.6 Discussion

With the high dimensional bosonization technology we recast the theory of fermionic fulids into a theory of bosons interacting under a Sine-Gordon type interaction vortex. The bosonized theory framework produces similar basic physics to the Hartree-Fock calculations. We showed that from reading off the low-energy effective bosonized action, we can see that it is odd in the order parameters and the **T** symmetry is broken spontaneously. The coefficients of the higher order terms suggest that the correction arises from them being coupled non-linearly to the curvature of the Fermi surfaces. Section 3.5.4 also suggests that by fixing the non-linear cosine scattering term to extremize the action, we can reproduce the behavior of the β_1 phase at the mean-field level. This suggests that there are two sources of non-linearity in the model. One is from the curvature that governs the Pomeranchuk-type physics, and the other is from the cosine term that governs the physics of the β_1 phase. The competition between these mechanisms is of great interest, since understanding this will help us go beyond the mean-field picture of the phase diagram. The stability condition for the bosonized theory is explored. We found that the stability condition is of the Pomeranchuk type. Furthermore for the ground state solutions to be stable, higher order dispersion terms are needed. Finally we compute self consistently the β_1. To classify the ground states further, we need to compute the Hall conductance. It turns out that the solution without the density condensate(Eq.(3.110)) gives vanishing Hall conductivity while the more reasonable ansatz does not give a consistent solution(Eq. (3.109)). It also important to note that this particular solution, Eq. (3.109) exhibits oscillation characteristic similar to charge-density wave.

Notice that the the order parameter that makes sense for α_1 phase is the density while for the β_1 phase, it is the phase difference($\Delta\phi_S$). This shows that bosonizing each band independently is not quite correct. The Hamiltonian we obtain does not know about the Berry curvature, which is more of a band-effect. The next step to move forward is to go back to the Fermionic picture to find an effective theory that depends on the Γ fields, which has bosonic statistics. In the next chapter, we implement this by carefully solving the electromagnetic-coupled fermionic theory. We are hoping that by doing it this way we can figure out what is the correct current to bosonize.

Chapter 4

Diagrammatic Approach and Effective Action

In this chapter we will show the derivation of the effective action for the two Fermi surface (two bands) model with quartic quadrupolar interactions. We will do our expansion from the symmetric phase. The calculation done in this chapter can be condered as an extension of a Hertz-Millis type effective theory([Hertz, 1976] and [Millis, 1993]). The basic of diagrammatic calculation and Green function calculation in the case of a spontaneous symmetry breaking can be reviewed at: [Fradkin, 2013], [Altland and Simons, 2010], [Mahan, 2013]. Curious readers can also find some inspiration in similar calculations that are done to various degrees of approximation in many systems ([Maciejko et al., 2013],[Fradkin et al., 2015], [Metzner et al., 1998], [Varma et al., 2002])

4.1 Loop Expansion and Skeleton Diagrams

We begin with the EM coupled action with the quadrupolar interactions. The free part of the action is taken to be

$$S_0[\psi^\dagger,\psi] = \int \mathrm{d}^3x\, \psi^\dagger_\alpha(x) \left[(D_0 + \varepsilon(-\mathbf{D}) - \varepsilon_F)\delta_{\alpha\beta} + v_F \Delta\sigma^z_{\alpha\beta}\right] \psi_\beta(x) \tag{4.1}$$

where $D_0 = \partial_0 - e\phi$, $D_a = \partial_a + ieA_a(\mathbf{x})$ with the Latin index $a = 1, 2$ enumerating the spatial coordinates. The Greek indices $\alpha, \beta = 1, 2$ are used to enumerate the different bands. The 3-vector $x = (x_0, \mathbf{x})$ is the 1+2 space-time vector and the action is written in the imaginary time or Euclidean signature; equivalently in the zero temperature ($T = 0$) limit. The spitting in energy between the bands is captured by the momentum parameter $\Delta > 0$ which leads to an energy difference of $2v_F\Delta$ between bands that is implemented by the last term in S_0. This splitting may be a result of either an applied Zeeman field or a Stoner instability towards a ferromagnetic phase. In the later case we consciously ignore the Goldstone modes (magnons) associated with that symmetry breaking. There may also other possible microscopic scenarios which lead to an isotropic split Fermi-surface which we will not delve too much into at the moment. The dispersion $\varepsilon(\mathbf{k})$ is taken to be isotropic in spatial directions.

The interaction part of the action is

$$S_{\text{int}}[\psi^\dagger, \psi] = \int \mathrm{d}^3x \int \mathrm{d}^3x', \left[\frac{f(\mathbf{x}-\mathbf{x}')}{2}\right] \delta(x_0 - x_0')\Phi_{\mu i}(x)\Phi_{\mu i}(x'), \quad \mu = x, y \quad i = 1, 2. \tag{4.2}$$

with the fermionic bilinears

$$\Phi_{\mu i}(x) := \psi_\alpha^\dagger(x)\mathcal{O}^{\mu i}_{\alpha\beta}(-i\mathbf{D})\psi_\beta(x) \tag{4.3}$$

$$\mathcal{O}^{\mu i}_{\alpha\beta}(-i\mathbf{D}) := \left(\frac{\sigma^\mu_{\alpha\beta}\tau^i_{ab}}{k_F^2}\right)(-i\overset{\leftrightarrow}{D}_a)(-i\overset{\leftrightarrow}{D}_b) \quad \text{or in k-space}$$

$$\mathcal{O}^{\mu i}_{\alpha\beta}(\mathbf{k})\ |_{\mathbf{A}=0} := \left(\frac{\sigma^\mu_{\alpha\beta}\tau^i_{ab}}{k_F^2}\right) k_a k_b \tag{4.4}$$

where the matrices $\tau^1 \equiv \sigma^z, \tau^2 \equiv \sigma^x$ are like the invariant tensors of the quadrupole channel viz. $\tau^1_{ab}p_ap_b = p_1^2 - p_2^2$ and $\tau^2_{ab}p_ap_b = 2p_ap_b$. The operator $\overset{\leftrightarrow}{\mathbf{D}} \equiv \overset{\leftrightarrow}{\nabla} + ie\mathbf{A} \equiv \frac{1}{2}(\overset{\rightarrow}{\nabla} - \overset{\leftarrow}{\nabla}) + ie\mathbf{A}$ is the symmetrized covariant derivative operator, which is a convenient convention for defining explicitly Hermitian velocity operators. The interaction kernel $f(\mathbf{x})$ is taken to be negative $f < 0$ for purposes of having attractive interactions. Note that $\Psi_{\mu i}$ defined this way is explicitly EM gauge invariant. For reference the explicit expansion of the "quadrupolar meson fields" in real-space is

$$\Phi_{\mu 1} = (-1)\left(\frac{\sigma^{\mu}_{\alpha\beta}}{k_F^2}\right)\left[\psi^{\dagger}_{\alpha}(\overleftrightarrow{\partial}_1 - \overleftrightarrow{\partial}_2)\psi_{\beta} + 2ie(A_1\psi^{\dagger}_{\alpha}\overleftrightarrow{\partial}_1\psi_{\beta} - A_2\psi^{\dagger}_{\alpha}\overleftrightarrow{\partial}_2\psi_{\beta}) - e^2(A_1^2 - A_2^2)\psi^{\dagger}_{\alpha}\psi_{\beta}\right] \tag{4.5}$$

$$\Phi_{\mu 2} = (-1)\left(\frac{\sigma^{\mu}_{\alpha\beta}}{k_F^2}\right)\left[2\psi^{\dagger}_{\alpha}\overleftrightarrow{\partial}_1\overleftrightarrow{\partial}_2\psi_{\beta} + 2ie(A_1\psi^{\dagger}_{\alpha}\overleftrightarrow{\partial}_2\psi_{\beta} + A_2\psi^{\dagger}_{\alpha}\overleftrightarrow{\partial}_1\psi_{\beta}) - 2e^2 A_1 A_2\psi^{\dagger}_{\alpha}\psi_{\beta}\right]. \tag{4.6}$$

Next a standard Hubbard-Stratonovich (HS) decoupling in the $\Phi_{\mu i}$ channel is applied to the interaction term. This produces the following HS decoupled action

$$S_{\text{tot}}[\psi^{\dagger}, \psi, \Gamma] = S_0[\psi^{\dagger}, \psi] + S_1[\Gamma] + S_2[\psi^{\dagger}, \psi, \Gamma] \tag{4.7}$$

$$S_0[\psi^{\dagger}, \psi] = \int \mathrm{d}^3x\ \psi^{\dagger}_{\alpha}\left[(D_0 + \varepsilon(-i\mathbf{D}) - \varepsilon_F)\delta_{\alpha\beta} + v_F\Delta\sigma^z_{\alpha\beta}\right]\psi_{\beta} \tag{4.8}$$

$$S_1[\Gamma] = \int \mathrm{d}^3x \int \mathrm{d}^3x' \left[\frac{-f^{-1}(\mathbf{x} - \mathbf{x}')}{2}\right]\delta(x_0 - x_0')\Gamma_{\mu i}(x)\Gamma_{\mu i}(x') \tag{4.9}$$

$$S_2[\psi^{\dagger}, \psi, \Gamma] = \int \mathrm{d}^3x\ \Phi_{\mu i}(x)\Gamma_{\mu i}(x) \tag{4.10}$$

where $\Gamma_{\mu i}(x)$ is the new HS field and

$$f^{-1}(\mathbf{r}) = \sum_{\mathbf{q}}\left[\frac{1}{f(\mathbf{q})}\right]\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}}, \quad f(\mathbf{r}) = \sum_{\mathbf{q}} f(\mathbf{q})\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}} \tag{4.11}$$

where an abuse of notation between the direct ($\mathbf{x}$) and Fourier space ($\mathbf{q}$) representations of the function f have been allowed to happen.

4.2 Integrating out fermions

The fermionic field ψ is integrated out to arrive at an effective action in terms of the HS fields. But first we expand the gauge fields (A_0, A_1, A_2) to second order which is same order as the interaction vertex $\Phi\Gamma$. Defining $\xi(\mathbf{k}) = \varepsilon(\mathbf{k}) - \varepsilon_F$ as the reduced energy. When minimally coupling in the dispersion $\nabla \to \nabla + ie\mathbf{A}$, there will be ordering ambiguities if the dispersion is not linear. This is settled by the standard Peierls substitution and semi-classical expansion up to second order in A. See Blount [Blount, 1962] for a more complete discussion. This introduces the velocity operator and the effective mass, which we take to be almost constant around the Fermi-energy. However it is crucial that the dispersion is non-quadratic or equivalently the DOS is non-constant and exhibits "band curvature". This is so that the free energy functional of the Γ theory can exhibit a second order phase transition. It was also demonstrated in the bosonized description that the higher-order terms are required for stability.

Thus the free Lagrangian $\mathcal{L}_0(\psi^\dagger, \psi, A)$ becomes

$$\begin{aligned}\mathcal{L}_0 =& \psi^\dagger_\alpha \partial_0 \psi_\alpha + \psi^\dagger_\alpha \xi(\mathbf{k}) \psi_\alpha + v_F \Delta \psi^\dagger_\alpha \sigma^z_{\alpha\beta} \psi_\beta - eA_0(x) \psi^\dagger_\alpha \psi_\alpha - \frac{ev_F}{2} \psi^\dagger_\alpha \left(\mathbf{A}(x) \cdot \hat{\mathbf{k}} + \hat{\mathbf{k}} \cdot \mathbf{A}(x) \right) \psi_\alpha \\ &+ \frac{e^2}{2m} |\mathbf{A}(x)|^2 \psi^\dagger_\alpha \psi_\alpha \end{aligned} \tag{4.12}$$

where the Fermi-velocity v_F and effective mass m are related to the isotropic dispersion $\xi(\mathbf{k})$ by

$$v_F \hat{k}_a = \partial_a \xi(\mathbf{k})|_{k_F}, \qquad \frac{\delta_{ab}}{m} = \partial^2_{ab} \xi(\mathbf{k})|_{k_F}$$

This gives "free" current density

$$\mathbf{j}_1 = \frac{\partial \mathcal{L}_0}{\partial \mathbf{A}} = \psi^\dagger_\alpha (v_F \hat{\mathbf{k}} + e\mathbf{A})\, \psi_\alpha \tag{4.13}$$

This is just the usual form of the current density, with the first term being the usual paramagnetic contribution and the second, the diamagnetic one. The quadrupolar interaction part of the action S_2, will also produce it's own current contribution which is

$$(\mathbf{j}_2)_a = \frac{\delta S_2}{\delta A_a} = -2ie\Gamma_{\mu i}\left(\frac{\sigma^{\mu}_{\alpha\beta}\tau^{i}_{ab}}{k_F^2}\right)\psi^{\dagger}_{\alpha}\overleftrightarrow{D}_b\psi_{\beta}. \tag{4.14}$$

The contribution S_1 does not produce a current density since the HS fields themselves $\Gamma_{\mu i}$ do not carry any EM charge. However the will develop couplings with $\mathbf{A}$ through the effective action after the fermionic field integration has been performed.

In summary the total action with $A = (A_0, \mathbf{A})$ explicitly is

$$S_{\mathrm{tot}}[\psi^{\dagger}, \psi, \Gamma, A] = S_0[\psi^{\dagger}, \psi, A] + S_1[\Gamma] + S_2[\psi^{\dagger}, \psi, \Gamma, A] \tag{4.15}$$

$$\begin{aligned} S_0[\psi^{\dagger}, \psi, A] = &\int \mathrm{d}^3x\, \psi^{\dagger}_{\alpha}(x)\left(\delta_{\alpha\beta}\,\xi(-i\nabla) + v_F\Delta\sigma^{z}_{\alpha\beta}\right)\psi_{\beta}(x) \\ &+ \int \mathrm{d}^3x(-e)A_0(x)\psi^{\dagger}_{\alpha}(x)\psi_{\alpha}(x) \\ &+ \int \mathrm{d}^3x\; ev_F A_a(x)\left[\psi^{\dagger}_{\alpha}(x)\left(\frac{\overleftrightarrow{\partial}_a}{|\nabla|}\right)\psi_{\alpha}(x)\right] \\ &+ \int \mathrm{d}^3x\left(\frac{e^2}{2m}\right)A_a(x)A_a(x)\psi^{\dagger}_{\alpha}(x)\psi_{\alpha}(x) \end{aligned} \tag{4.16}$$

$$S_1[\Gamma] = \int \mathrm{d}^3x \int \mathrm{d}^3x\left[\frac{-f^{-1}(\mathbf{x}-\mathbf{x}')}{2}\right]\delta(x_0 - x_0')\Gamma_{\mu i}(x)\Gamma_{\mu i}(x') \tag{4.17}$$

$$\begin{aligned} S_2[\psi^{\dagger}, \psi, \Gamma, A] = &\int \mathrm{d}^3x\,\Gamma_{\mu i}(x)\left(\frac{-\sigma^{\mu}_{\alpha\beta}\tau^{i}_{ab}}{k_F^2}\right)\left[\psi^{\dagger}_{\alpha}(x)\overleftrightarrow{\partial}_a\overleftrightarrow{\partial}_b\psi(x)\right] \\ &+ \int \mathrm{d}^3x\Gamma_{\mu i}(x)\left(\frac{-2ie\sigma^{\mu}_{\alpha\beta}\tau^{i}_{ab}}{k_F^2}\right)A_a(x)\left[\psi^{\dagger}_{\alpha}(x)\overleftrightarrow{\partial}_b\psi_{\beta}(x)\right] \\ &+ \int \mathrm{d}^3\,\Gamma_{\mu i}(x)\left(\frac{e^2\sigma^{\mu}_{\alpha\beta}\tau^{i}_{ab}}{k_F^2}\right)A_a(x)A_b(x)\psi^{\dagger}_{\alpha}(x)\psi_{\beta}(x) \end{aligned} \tag{4.18}$$

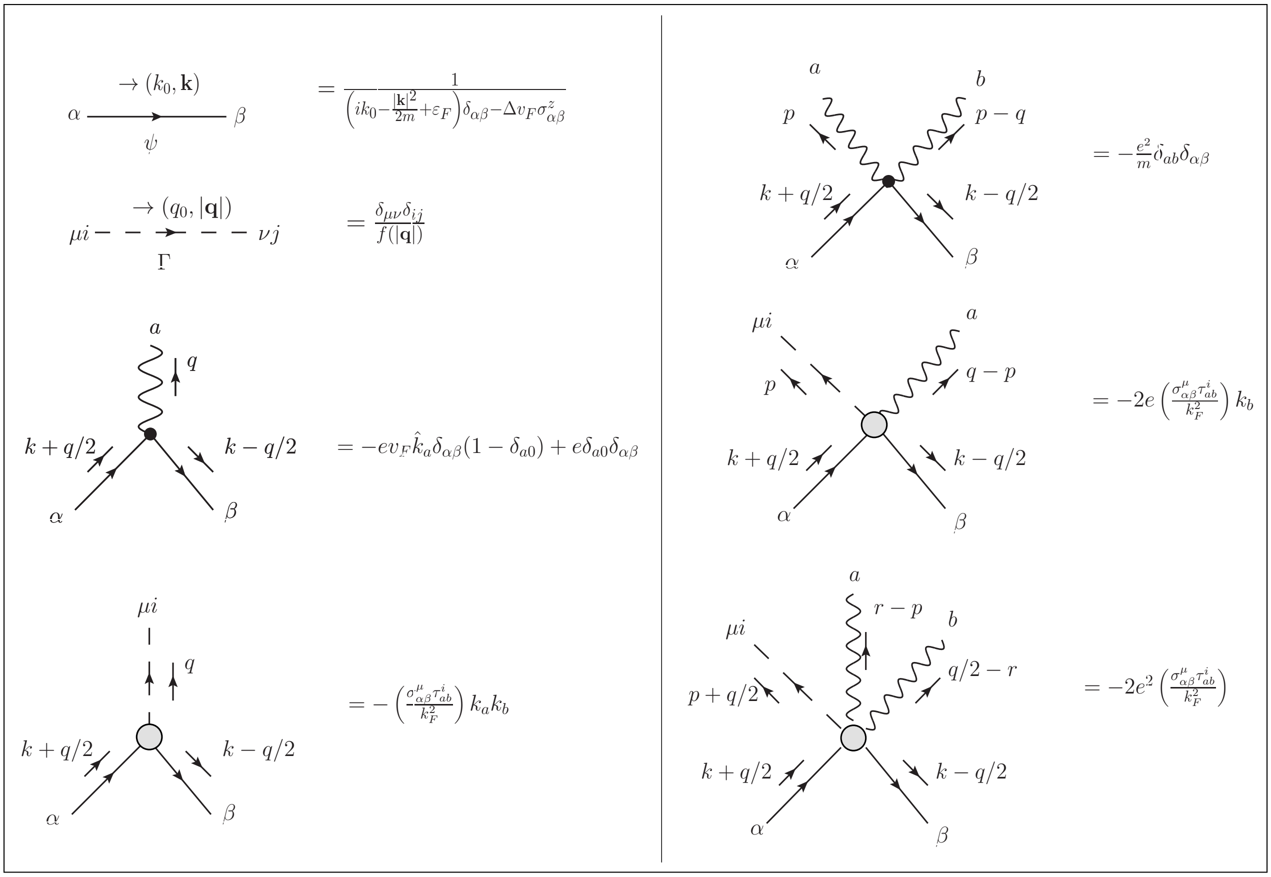


Figure 4.1: Diagramatic lines and fermions for the Feynman rules. There is also the obligatory $e^{ik_0 \cdot 0^+}$ and (-1) sign within fermion loops $\int \frac{dk_0}{(2\pi)}$ which are non-propagating.

From this total gauge invariant action, we can read off the various propagators lines and interaction vertices in $(k_0, \mathbf{k})-$space. These are listed in Fig. 4.1.

The total partition function is then

$$Z_{\text{tot}} = \int d[\Gamma] \int d[\psi, \psi^\dagger] \exp\left(-S_0[\psi^\dagger, \psi, A] - S_1[\Gamma] - S_2[\psi^\dagger, \psi, \Gamma, A]\right). \tag{4.19}$$

Notationally we can compactly express the action by suppressing the integrations and contraction of indices, where the interpretation is obvious. So we have

$$S_0[\psi^\dagger,\psi,A] = -\psi^\dagger\hat{\mathcal{G}}_0^{-1}\psi - eA_0\psi^\dagger\psi + ev_F A\psi^\dagger\hat{\mathbf{k}}\psi + \tfrac{e^2}{2m}A^2\psi^\dagger\psi \tag{4.20}$$

$$S_1[\Gamma] = -\frac{1}{2f}\Gamma\Gamma \tag{4.21}$$

$$S_2[\psi^\dagger,\psi,\Gamma,A] = \psi^\dagger\hat{\mathcal{O}}_2\psi\Gamma + 2eA\psi^\dagger\hat{\mathcal{O}}_1\psi\Gamma - e^2A^2\hat{\mathcal{O}}_0\psi^\dagger\psi\Gamma \tag{4.22}$$

where $\hat{\mathcal{O}}_n$ is the vertex operator with n internal momenta operators $-i\overleftrightarrow{\partial}$ contracted with the quadrupolar coupling constant $\sigma\tau/(k_F)^2$. Then formally integrating ψ first we get the loop expansion as

$$\begin{aligned}
Z_{\text{tot}} =& \int d[\Gamma]\mathrm{e}^{-S_1[\Gamma]} \det\left[\frac{\delta^2}{\delta\psi\delta\psi^\dagger}\left(S_0[\psi^\dagger,\psi,A] + S_2[\psi^\dagger,\psi,\Gamma,A]\right)\right] \\
=& \int d[\Gamma]\mathrm{e}^{-S_1[\Gamma]}\exp\left\{\mathrm{Tr}\ln\left[-\hat{\mathcal{G}}_0^{-1} + \hat{\mathcal{O}}_2\Gamma - eA_0 + ev_FA\hat{\mathbf{k}} + \tfrac{e^2}{2m}A^2 + 2eA\hat{\mathcal{O}}_1\Gamma - e^2A^2\hat{\mathcal{O}}_0\Gamma\right]\right\} \\
=& Z_{\psi_0}\int d[\Gamma]\mathrm{e}^{-S_1[\Gamma]}\exp\left\{\mathrm{Tr}\ln\Big(\hat{1} - \hat{\mathcal{G}}_0\hat{\mathcal{O}}_2\Gamma + e\hat{\mathcal{G}}_0A_0 - ev_FA\hat{\mathcal{G}}_0\hat{\mathbf{k}} - \tfrac{e^2}{2m}A^2\hat{\mathcal{G}}_0 - 2eA\hat{\mathcal{G}}_0\hat{\mathcal{O}}_1\Gamma\right. \\
&\left. + e^2A^2\hat{\mathcal{G}}_0\hat{\mathcal{O}}_0\Gamma\Big)\right\}
\end{aligned} \tag{4.23}$$

where Z_{ψ_0} is the free fermion partition function, ie. $\det[-\hat{\mathcal{G}}_0^{-1}]$. Next focusing only on the factors dependent on Γ, we determine an effective action for Γ. Expanding to one loop order (single trace in fermion lines) , we get terms in the exponential (effective action) that appear as

$$
\begin{aligned}
S_{\text{eff}}[\Gamma] = &-\frac{1}{2f}\Gamma\Gamma - \mathrm{Tr}\left(-\hat{\mathcal{G}}_0\hat{\mathcal{O}}_2\Gamma + eA_0\hat{\mathcal{G}}_0 - ev_F A\hat{\mathcal{G}}\hat{\mathbf{k}} - \tfrac{e^2A^2}{2m}\hat{\mathcal{G}}_0 - 2eA\hat{\mathcal{G}}_0\hat{\mathcal{O}}_1\Gamma + e^2A^2\hat{\mathcal{G}}_0\hat{\mathcal{O}}_0\Gamma\right)\\
&+ \tfrac{1}{2}\mathrm{Tr}\left([-\hat{\mathcal{G}}_0\hat{\mathcal{O}}_2\Gamma + \ldots]^2\right) \quad \text{2nd order}\\
&- \tfrac{1}{3}\mathrm{Tr}\left([-\hat{\mathcal{G}}_0\hat{\mathcal{O}}_2\Gamma + \ldots]^3\right) \quad \text{3rd order}\\
&+ \tfrac{1}{4}\mathrm{Tr}\left([-\hat{\mathcal{G}}_0\hat{\mathcal{O}}_2\Gamma + \ldots]^4\right) \quad \text{4th order}\\
&- \ldots
\end{aligned}
\tag{4.24}
$$

The first line is recognized as the mean-field terms with the Hubbard-Stratonovich decoupling term that functions as a potential for the bosonic Γ field. The 2nd order expansion should give the lowest order self-energy corrections to the fermions and their contributions to the effective action for Γ. But there are still quite a few terms to consider. The simplest rule that can be used to simplify the book-keeping is to note that because the $\hat{\mathcal{O}}_n$ vertices are "flavor" changing on the fermion lines, that it is a requirement that $\hat{\mathcal{O}}_n$ appear an even number of times in a fermion loop. The other symmetry constraints require more analysis.

4.3 The free fermion partition function Z_{ψ_0} and gapless-ness

There is always a clear and present danger whenever one attempts at integrating out fermions in a gapless phase. The astute reader would object that the free fermion partition function factor Z_{ψ_0} given by the following functional determinant

$$
Z_{\psi_0} = \det[-\hat{\mathcal{G}}_0^{-1}] = 0 \tag{4.25}
$$

is actually *zero* when the free Hamiltonian, $\hat{H}_0 = -\hat{\mathcal{G}}_0^{-1} + \partial_\tau$ is gapless. That is the operator $\hat{\mathcal{G}}_0^{-1}$ has a non-trivial kernel in the zero temperature limit. But in the limit of finite temperature, the spectrum of $\hat{\mathcal{G}}_0^{-1}$ has an IR cutoff, set by the smallest Matsubara frequency $\omega_1 = \pi/\beta = \pi k_B T$. Thus we must interpret our previous functional manipulations as being IR regulated by some effective low temperature whose limit we take to zero at the end of our calculations.

With this caveat in mind, the sole purpose of the previous formal manipulations is to arrive at a new gapless phase for the ψ field when the Γ fields are slowly varying and treated in an adiabatic (stationary phase) approximation. Thus so we should not be thinking of integrating it out ψ entirely by dropping the Z_{ψ_0}. But if instead the classical field Γ_{class} extremizes the effective action S_{eff} in the stationary phase approximation then , suppressing $A = 0$ for the moment, we have

$$
\begin{aligned}
Z_{\text{tot}} &= \int d[\Gamma]\, \mathrm{e}^{-S_1[\Gamma]} \det\left[-\hat{\mathcal{G}}_0^{-1} + \hat{\mathbb{O}}_2\Gamma\right] \\
&= Z_{\psi_0} \int d[\Gamma]\ \exp\left(-S_1[\Gamma] + \operatorname{Tr}\ln\left[\hat{1} - \hat{\mathcal{G}}_0^{-1}\hat{\mathbb{O}}_2\Gamma\right]\right) \\
&= Z_{\psi_0} \int d[\Gamma]\ \exp\left(-S_{\text{eff}}[\Gamma]\right) \\
&= Z_{\psi_0} \int d[\delta\Gamma]\ \exp\left(-S_{\text{eff}}[\Gamma_{\text{class}} + \delta\Gamma]\right) \\
&\approx Z_{\psi_0} \int d[\delta\Gamma]\ \exp\left(-S_{\text{eff}}[\Gamma_{\text{class}}] - S''_{\text{eff}}[\Gamma_{\text{class}}]\delta\Gamma^2\right) \\
&= Z_{\psi_0} \exp\left(-S_{\text{eff}}[\Gamma_{\text{class}}]\right) \int d[\delta\Gamma] \exp\left(-S''_{\text{eff}}[\Gamma_{\text{class}}]\delta\Gamma^2\right) \\
&= Z_{\psi_0} \exp\left(-S_1[\Gamma_{\text{class}}] + \operatorname{Tr}\ln\left[\hat{1} - \hat{\mathcal{G}}_0^{-1}\hat{\mathbb{O}}_2\Gamma_{\text{class}}\right]\right) Z_{\delta\Gamma,\Gamma_{\text{class}}} \\
&= \det\left[-\hat{\mathcal{G}}_0^{-1} + \hat{\mathbb{O}}_2\Gamma_{\text{class}}\right] \mathrm{e}^{-S_1[\Gamma_{\text{class}}]} Z_{\delta\Gamma,\Gamma_{\text{class}}} \\
&= Z_{\delta\Gamma,\Gamma_{\text{class}}} \mathrm{e}^{-S_1[\Gamma_{\text{class}}]} \int d[\psi,\psi^\dagger]\ \exp\left(-\psi^\dagger[-\hat{\mathcal{G}}_0^{-1} + \hat{\mathbb{O}}_2\Gamma_{\text{class}}]\psi\right)
\end{aligned}
$$

where $Z_{\delta\Gamma,\Gamma_{\text{class}}}$ is the partition function obtained from integrating the Gaussian fluctuations about Γ_{class}. Thus the symmetry broken phase, to zeroth order in $\delta\Gamma$ is that of a gapless ψ

phase with the following mean-field action

$$S_{\mathrm{MF}}[\psi, \psi^\dagger; \Gamma_{\mathrm{class}}] = -\psi^\dagger \hat{\mathcal{G}}_0^{-1} \psi + \Gamma_{\mathrm{class}} \psi^\dagger \hat{\mathbb{O}}_2 \psi. \tag{4.26}$$

4.4 Landau-Ginzburg vs Self Consistent Mean-field

Lastly there is a subtle difference between self-consistent mean-fields and Landau free energy functionals obtained by low-order expansions of S_{eff}. Now the exact effective action $S_{\mathrm{eff}}[\Gamma]$ is given by the trace log expression

$$S_{\mathrm{eff}}[\Gamma] = S_1[\Gamma] - \mathrm{Tr} \ln \left[\hat{1} - \hat{\mathcal{G}}_0^{-1} \hat{\mathbb{O}}_2 \Gamma \right] \tag{4.27}$$

and the self-consistent method is equivalent to extremizing this functional on Γ which automatically enforces self-consistency. This is in general a non-analytic functional on Γ near the second-order phase transition, as it has to be. However a Landau expansion to $O(\Gamma^4)$, which assumes analyticity of the functional generally produces different minima since all the higher order terms have been neglected. It can only be expected that near the critical point that the exact and approximate method by first expanding S_{eff} will produce similar minima. The expansion of S_{eff}, though probably only valid near the phase transition, however has the two advantages. And that is, its couplings can be traced back to specific loop diagrams and is independent of a mean-field ansatz.

4.5 The Effective Action and Free Energy

The form of $S_{\text{eff}}[\Gamma]$ can be determined from purely symmetry arguments. Note that the global symmetry of the full fermionic action is $\mathbb{Z}_2\times$U(1)$\times$O(2)$\cong$ O(2)$\times$O(2). The unitary U(1) subgroup stems from the *relative phase* of the ψ_1, ψ_2 fermions and is generated by the isospin rotation $\mathrm{e}^{i\varphi\sigma^z/2}$ acting globally. The orbital or isospin symmetry was broken from U(2) to U(1) due to the $\Delta\sigma^z$ term in the free part of the Hamiltonian. There is also the gauged U(1) or total phase of ψ_1 and ψ_2 which is just normal $\mathrm{U}(1)_{\mathrm{EM}}$ gauge symmetry which is distinct from the global isospin symmetry. The O(2) is the spatial rotational and spatial mirror symmetry. Lastly $\mathbb{Z}_2$ is the anti-linear time-reversal or complex conjugation. This is the "mirror" associated to the relative phase U(1) group since it maps $\{\sigma^x, \sigma^y, \sigma^z\}$ to $\{\sigma^x, -\sigma^y, \sigma^z\}$ which is symmetry of the microscopic Hamiltonian.

Having identified the group O(2)$\times$O(2) it is clear what the transformation properties of $\Phi_{\mu i}$ are.

U(1): $\Phi_{\mu i} \mapsto S(\varphi)_{\mu\nu}\Phi_{\nu i}$ where $S(\varphi) = \exp(i\varphi\sigma^z/2) \in \mathrm{SU}(2)$

O(2): $\Phi_{\mu i}(\mathbf{k}) \mapsto R(2\vartheta)_{ij}\Phi_{\mu j}(R^T(\vartheta)\cdot\mathbf{k})$ where $R(2\vartheta), R(\vartheta) \in$O(2) are induced by either a proper rotation $\theta_{\mathbf{k}} \mapsto \theta_{\mathbf{k}} + \vartheta$, a reflection $\theta_{\mathbf{k}} \mapsto -\theta_{\mathbf{k}}$ or both.

$\mathbb{Z}_2$: $[\Phi_{xi}(\mathbf{r}), \Phi_{yi}(\mathbf{r})] \mapsto [\Phi_{xi}(\mathbf{r}), -\Phi_{yi}(\mathbf{r})]$ and in momentum space
$[\Phi_{xi}(\mathbf{k}), \Phi_{yi}(\mathbf{k})] \mapsto [\Phi_{xi}(-\mathbf{k}), -\Phi_{yi}(-\mathbf{k}])$

The effective action is constructed from symmetric polynomials of $\Gamma_{\mu i}$ contracted with invariant tensors of O(2)$\times$O(2). The invariant tensors of a factor SO(2) $\cong$U(1) are δ_{ij} and ϵ_{ij} and powers thereof. These second rank tensors correspond to taking the dot($\cdot$) and cross ($\times$) products on 2-vectors. While δ_{ij} is mirror invariant, $\epsilon_{ij} \to -\epsilon$ under mirror. From these second rank tensors, the first few invariant tensors of SO(2)×SO(2) are $\delta_{\mu\nu}\delta_{ij}$, $\epsilon_{\mu\nu}\epsilon_{ij}$ and so on. Written out, some of these are

$$\delta_{\mu\nu}\delta_{ij}\Gamma_{\mu i}\Gamma_{\nu j} = |\vec{\Gamma}_1|^2 + |\vec{\Gamma}_2|^2 \tag{4.28}$$

$$\epsilon_{\mu\nu}\epsilon_{ij}\Gamma_{\mu i}\Gamma_{\nu j} = 2\,\vec{\Gamma}_1 \times \vec{\Gamma}_2 \tag{4.29}$$

where the overhead $\vec{*}$ corresponds to the iso-spin x, y direction. So for example we can expect the terms at second order of the 1-loop calculation to yield

$$I^{(2)}_{\mu i,\nu j}\Gamma_{\mu i}\Gamma_{\nu j} = a(|\vec{\Gamma}_1|^2 + |\vec{\Gamma}_2|^2) + 2b(\vec{\Gamma}_1 \times \vec{\Gamma}_2) \tag{4.30}$$

where $a = I^{(2)}_{x1,x1} = I^{(2)}_{y1,y1} = I^{(2)}_{x2,x2} = I^{(2)}_{y2,y2}$ and $b = I^{(2)}_{x1,y2} = -I^{(2)}_{y1,x2} = I^{(2)}_{y2,x1} = -I^{(2)}_{x2,y1}$. But $b = 0$ from the the O(2) symmetry.

As a warm-up, we focus first on the static and homogeneous parts of the effective energy. The so-called 1-loop ground state energy or free energy term. This is essentially the calculation done in Ref. [Sun and Fradkin, 2008]. The terms of interest originate from the expansion of Eq. (4.24) in the 2nd order and fourth order.

4.5.1 Quadratic interaction of Γ

The simplest diagram to consider is the following $O(\Gamma^2)$ diagram.

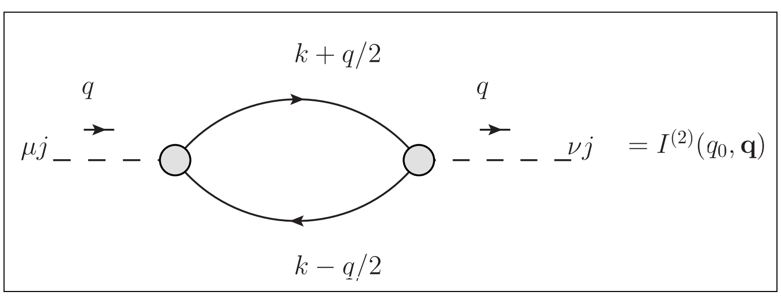


Figure 4.2: The mass bubble for the $\Gamma_{\mu i}$ field.

Applying the standard rules gives

$$I^{(2)}_{\mu i\nu j}(q) = \frac{(-1)}{k_F^4}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\,\frac{1}{\beta}\sum_{k_0}(\mathbf{k}^T\tau^i\mathbf{k})(\mathbf{k}^T\tau^j\mathbf{k})\sum_{mn}\sigma^\mu_{mn}\sigma^\nu_{nm}G_m(ik_0^+,\mathbf{k}^+)G_n(ik_0^-,\mathbf{k}^-)$$

In the static limit we set $q=0$ and the straightforwardly get

$$I^{(2)}_{\mu i\nu j}(q=0) = \frac{2\delta_{\mu\nu}}{k_F^4}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\,(\mathbf{k}^T\tau^i\mathbf{k})(\mathbf{k}^T\tau^j\mathbf{k})\left(\frac{n_F(\xi_{2\mathbf{k}})-n_F(\xi_{1\mathbf{k}})}{2v_F\Delta}\right)$$

after a the $\frac{1}{\beta}\sum_{k_0}(*)$ summation and using the handy relations $\sigma^\mu_{12}\sigma^\nu_{21} = \delta_{\mu\nu}+i\epsilon_{\mu\nu}$ and $\sigma^\mu_{21}\sigma^\nu_{12} = \delta_{\mu\nu}-i\epsilon_{\mu\nu}$. The remaining integral is carried out by linearizing the momentum $\mathbf{k}\approx k_F\hat{\mathbf{k}}$ and performing the angular $\int\frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi}$ and radial integral $\int\mathrm{d}|\mathbf{k}|$ separately. The radial momentum integral is performed by transforming to the reduced energy $\xi_{\mathbf{k}}$ by use of the DOS $N(\xi)$ factor. Specifically the change of measure is given by $\int\frac{\mathrm{d}^2k}{(2\pi)^2}\rightarrow\int\frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi}\int\mathrm{d}\xi\,N(\xi)$. This yields

$$I^{(2)}_{\mu i\nu j}(q=0) = r\delta_{\mu\nu}\delta_{ij} \tag{4.31}$$

where we have defined the averaged DOS quantity

$$r := \frac{1}{2v_F\Delta}\int_{-v_F\Delta}^{v_F\Delta}N(\xi)\mathrm{d}\xi \approx N(0)+\frac{(v_F\Delta)^2}{6}N''(0). \tag{4.32}$$

where the approximation comes from making a Taylor approximation on $N(\xi)$ about the Fermi-energy $\xi=0$.

Still working in the limit of no external fields $A\equiv 0$, we proceed next to compute the gradient contributions to the effective action from the same bubble diagram Fig. 4.2 but now with small but non-zero q. The static and homogeneous $(q_0,\mathbf{q})=(0,0)$ contributions has been previously computed to give the “mass" to Γ. The part of the action $S_1[\Gamma]$ contains already a "bending energy" term in the form of

$$\begin{aligned}\mathcal{L}_1 &= -\frac{1}{2f(\mathbf{q})}\Gamma(-q)_{\mu i}\Gamma(q)_{\mu i} \\ &= \frac{1}{2|f(0)|}\Gamma_{\mu i}(-q)\Gamma_{\mu i}(q) + \frac{\kappa|\mathbf{q}|^2}{2}\Gamma_{\mu i}(-q)\Gamma_{\mu i}(q)\end{aligned} \tag{4.33}$$

where we use the following form of the interaction kernel

$$f(\mathbf{q}) = \frac{f(0)}{1 + f(0)\kappa|\mathbf{q}|^2}. \tag{4.34}$$

and $f(0) < 0$ in the attractive case. The static but non-homogeneous gradient $q_0 = 0, \mathbf{q} \neq 0$ merely produces a renormalization to this term, or equivalently to κ. To lowest order, we make the approximation that neglects this small renormalization and proceed to compute the dynamical $q_0 \neq 0$ corrections. Alternatively, as we show in Appendix A.6, the higher order spatial gradients generally come with irrelevant coupling constants.

To arrive at the gradient expansion terms in Eq. (4.31), we use our gradient expansion technique described in Appendix A.4. Applying the relation [F] of Appendix A.4.2 yield from the summation of channels

$$\begin{aligned}\sum_{mn}\sigma^{\mu}_{mn}\sigma^{\nu}_{nm}G_m(ik_0^+,\mathbf{k}^+)G_n(ik_0^-,\mathbf{k}^-) =& 2\delta_{\mu\nu}G_1(k)G_2(k) + i\epsilon_{\mu\nu}\left[\frac{v_F\delta q}{2v_F\Delta}(G_1(k)-G_2(k))^2\right. \\ &\left. + \left(\frac{iq_0}{v_F\Delta}\right)G_1(k)G_2(k)\right]\end{aligned}$$

where $\delta q := \mathbf{q}\cdot\hat{\mathbf{k}}$. Next we linearize $k_a \approx k_F\hat{k}_a$ and perform the angular and radial integral with the DOS thus giving

$$I^{(2)}_{\mu i\nu j}(q) = r\delta_{ij}\delta_{\mu\nu} + \left(\frac{r}{2v_F\Delta}\right)(\delta_{ij}i\epsilon_{\mu\nu})\, iq_0 \tag{4.35}$$

which is valid to leading order in the $O(q)$ expansion, with $\frac{\|q\|}{v_F\Delta}$ taken to be the small parameter where $\|q\| := \sqrt{v_F^2|\mathbf{q}|^2 + q_0^2}$. The $v_F\delta q$ term does not survive the angular integral integral because of the uncompensated angular harmonics. Also the identity $G_1G_2 = (G_1 - G_2)/(\xi_1 - \xi_2)$ was used.

A higher order q expansion is presented in Appendix A.6 where there are at second order, spatial $|\mathbf{q}|^2$ terms which renormalize κ. But as a simplest approximation we will neglect this small correction or absorb it into a redefinition of κ.

The contribution to the effective Lagrangian appears as

$$\begin{aligned}\delta\mathcal{L}^{(2)}_{\text{eff}}(q) &= -\tfrac{1}{2}\, I^{(2)}_{\mu i\nu j}(q)\Gamma_{\mu i}(q)\Gamma_{\nu j}(-q)\\ &= -\frac{r}{2}\Gamma_{\mu i}(q)\Gamma_{\mu i}(-q) - \frac{1}{2}\left(\frac{r}{2v_F\Delta}\right) iq_0\left[\delta_{ij}i\epsilon_{\mu\nu}\Gamma_{\mu i}(q)\Gamma_{\nu j}(-q)\right]\end{aligned}$$

or in real space

$$\delta\mathcal{L}^{(2)}_{\text{eff}} = -\frac{r}{2}\Gamma_{\mu i}(\mathbf{x},\tau)\Gamma_{\mu i}(\mathbf{x},\tau) - \frac{i}{2}\left(\frac{r}{2v_F\Delta}\right)\epsilon_{\mu\nu}\Gamma_{\mu i}(\mathbf{x},\tau)\partial_\tau\Gamma_{\nu i}(\mathbf{x},\tau)$$

This final expression is in fact O(2)×O(2) invariant despite the appearance of the $\epsilon_{\mu\nu}$ factor which is only SO(2) invariant under band or orbital transformations and odd under time-reversal. This is because the $q_0 \equiv i\partial_\tau \equiv \partial_t$ is (in real-time) odd under time-reversal and compensates for the $\epsilon_{\mu\nu} \to -\epsilon_{\mu\nu}$ under time-reversal.

It is peculiar that in the limit $|\mathbf{q}| \ll \Delta$, the gradient terms above do not reduce to the familiar Hertz-Millis form $\propto iq_0/(v_F|\mathbf{q}|)$. Also the $\Delta = 0$ limit is singular because the Δ's appear in the denominators. This can be understood from the observation that despite have a gapless Fermi-liquid spectrum, the interactions mediated by the Hubbard-Stratonovich field Γ near $\mathbf{q} = 0$ are gapped with a characteristic energy of $v_F \Delta$. Thus it is not too surprising that the coefficients of the effective action obtained by integrating out these gapped ψ fluctuations appear with powers of Δ in the denominator.

4.5.2 Quartic interaction of Γ

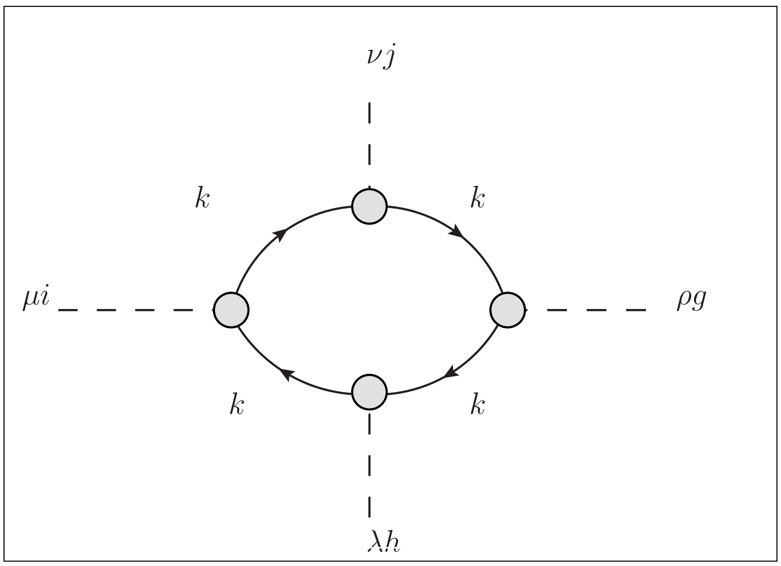


Figure 4.3: The quartic interaction bubble for the $\Gamma(q = 0)_{\mu i}$ stationary-homogeneous mode.

The next term to consider is the quartic interaction of Γ shown in Fig. 4.3, specialized to the $q = 0$ limit. Again applying the rules and after computing the internal $\int \mathrm{d}k_0/(2\pi)$ integral in the $q = 0$ limit yields after linearizing $\mathbf{k} \approx k_F k \hat{\mathbf{k}}$

$$I^{(4)}_{\mu i,\nu j,\sigma g,\lambda h} = -\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{\beta} \sum_{k_0} \sum_{m,n,l,o} e_i(\theta_{\mathbf{k}}) e_j(\theta_{\mathbf{k}}) e_g(\theta_{\mathbf{k}}) e_h(\theta_{\mathbf{k}}) \sigma^{\mu}_{mn} \sigma^{\lambda}_{nl} \sigma^{\rho}_{lo} \sigma^{\nu}_{om} \times \left(\frac{1}{ik_0 - \xi_{m\mathbf{k}}}\right)\left(\frac{1}{ik_0 - \xi_{n\mathbf{k}}}\right)\left(\frac{1}{ik_0 - \xi_{l\mathbf{k}}}\right)\left(\frac{1}{ik_0 - \xi_{o\mathbf{k}}}\right). \tag{4.36}$$

where $e_i(\theta_{\mathbf{k}}) := \hat{\mathbf{k}}^T \tau^i \hat{\mathbf{k}}$. We expect that $m = l, n = 0$ and so we resolve the following fractions accordingly

$$\left(\frac{1}{ik_0-\xi_{m\mathbf{k}}}\right)\left(\frac{1}{ik_0-\xi_{n\mathbf{k}}}\right)\equiv\left(\frac{1}{\xi_{n\mathbf{k}}-\xi_{m\mathbf{k}}}\right)\left(\frac{1}{ik_0-\xi_{n\mathbf{k}}}-\frac{1}{ik_0-\xi_{m\mathbf{k}}}\right)$$
$$\left(\frac{1}{ik_0-\xi_{l\mathbf{k}}}\right)\left(\frac{1}{ik_0-\xi_{o\mathbf{k}}}\right)\equiv\left(\frac{1}{\xi_{o\mathbf{k}}-\xi_{l\mathbf{k}}}\right)\left(\frac{1}{ik_0-\xi_{o\mathbf{k}}}-\frac{1}{ik_0-\xi_{l\mathbf{k}}}\right)$$

giving for the Matsubara frequency sum

$$\left(\frac{1}{\xi_{n\mathbf{k}}-\xi_{m\mathbf{k}}}\right)\left(\frac{1}{\xi_{o\mathbf{k}}-\xi_{l\mathbf{k}}}\right)\frac{1}{\beta}\sum_{k_0}\Bigg\{\left(\frac{1}{ik_0-\xi_{n\mathbf{k}}}\right)\left(\frac{1}{ik_0-\xi_{o\mathbf{k}}}\right)-\left(\frac{1}{ik_0-\xi_{n\mathbf{k}}}\right)\left(\frac{1}{ik_0-\xi_{l\mathbf{k}}}\right)$$
$$-\left(\frac{1}{ik_0-\xi_{m\mathbf{k}}}\right)\left(\frac{1}{ik_0-\xi_{o\mathbf{k}}}\right)+\left(\frac{1}{ik_0-\xi_{m\mathbf{k}}}\right)\left(\frac{1}{ik_0-\xi_{l\mathbf{k}}}\right)\Bigg\}.$$

Now from the matrix elements, we know that $\xi_{n\mathbf{k}}=\xi_{o\mathbf{k}}\neq\xi_{m\mathbf{k}}=\xi_{l\mathbf{k}}$ for $\Delta>0$. So have double poles in the analytic continuation during the Matsubara sum. Performing the contour integral yields the above expression as

$$\left(\frac{1}{\xi_{n\mathbf{k}}-\xi_{m\mathbf{k}}}\right)\left(\frac{1}{\xi_{o\mathbf{k}}-\xi_{l\mathbf{k}}}\right)\oint_{-\mathcal{C}}\frac{\mathrm{d}z}{i2\pi}\left\{\frac{1}{(z-\xi_{n\mathbf{k}})^2}-\frac{2}{(z-\xi_{m\mathbf{k}})(z-\xi_{n\mathbf{k}})}+\frac{1}{(z-\xi_{m\mathbf{k}})^2}\right\}n_F(z)$$
$$=\left(\frac{1}{\xi_{n\mathbf{k}}-\xi_{m\mathbf{k}}}\right)\left(\frac{1}{\xi_{o\mathbf{k}}-\xi_{l\mathbf{k}}}\right)\left\{n_F'(\xi_{n\mathbf{k}})+n_F'(\xi_{m\mathbf{k}})-2\left[\frac{n_F(\xi_{m\mathbf{k}})-n_F(\xi_{n\mathbf{k}})}{\xi_{m\mathbf{k}}-\xi_{n\mathbf{k}}}\right]\right\}$$

Next we sum over all channels and then perform the $\int\mathrm{d}\xi N(\xi)$ integral.

$$I^{(4)}_{\mu i,\nu j,\rho g,\lambda h}=4\left(\delta_{\mu\lambda}\delta_{\rho\nu}-\epsilon_{\mu\lambda}\epsilon_{\rho\nu}\right)\left[\frac{\bar{r}-r}{(2v_F\Delta)^2}\right]\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi}e_i(\theta_{\mathbf{k}})e_j(\theta_{\mathbf{k}})e_g(\theta_{\mathbf{k}})e_h(\theta_{\mathbf{k}})$$
$$=\tfrac{1}{2}\left(\delta_{\mu\lambda}\delta_{\rho\nu}-\epsilon_{\mu\lambda}\epsilon_{\rho\nu}\right)\left(\delta_{ij}\delta_{gh}+\delta_{ig}\delta_{jh}+\delta_{ih}\delta_{jg}\right)\left[\frac{\bar{r}-r}{(2v_F\Delta)^2}\right]$$

where we have used the following easily checked relations

$$\sigma^{\mu}_{12}\sigma^{\nu}_{21} = \delta_{\mu\nu} + i\epsilon_{\mu\nu}, \qquad \sigma^{\mu}_{21}\sigma^{\nu}_{12} = \delta_{\mu\nu} - i\epsilon_{\mu\nu}$$

$$\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi} e_i(\theta_{\mathbf{k}})e_j(\theta_{\mathbf{k}})e_g(\theta_{\mathbf{k}})e_h(\theta_{\mathbf{k}}) = \frac{3}{8}\delta_{(ij}\delta_{gh)} = \frac{1}{8}\left(\delta_{ij}\delta_{gh} + \delta_{ig}\delta_{jh} + \delta_{ih}\delta_{jg}\right)$$

and have defined the mean DOS quantity

$$\bar{r} = \frac{N(v_F\Delta) + N(-v_F\Delta)}{2} \approx N(0) + \frac{(v_F\Delta)^2}{2}N''(0) \tag{4.37}$$

at the Fermi-surfaces $\xi = \pm v_F\Delta$.

Thus we need non-trivial band curvature $r \neq \bar{r}$ ie varying DOS for this coefficient (coupling constant) to be non-zero. This agrees with the earlier mean-field and bosonization studies of nematic metals. This diagram then leads to the following contribution to the effective Lagrangian

$$\begin{aligned}\mathcal{L}^{(4)}_{\text{eff}} &= -\frac{1}{4} I_{\mu i,\nu j,\rho g,\lambda h}\Gamma_{\mu i}\Gamma_{\nu j}\Gamma_{\rho g}\Gamma_{\lambda h} \\ &= \frac{1}{8}\left[\frac{r-\bar{r}}{(2v_F\Delta)^2}\right]\left[(\Gamma_{\mu i}\Gamma_{\mu i})^2 + 2\left(\Gamma_{\mu i}\Gamma_{\mu j}\Gamma_{\nu j}\Gamma_{\nu i}\right)\right]\end{aligned}$$

which is manifestly O(2)×O(2) invariant. Now Taylor expanding the DOS gives

$$r - \bar{r} \approx -\frac{N''(0)}{12}(2v_F\Delta)^2$$

which is required to be positive for stability. However as we will discuss in the next subsection, that fixing the density may produce a positive term at similar order which aid in stability. We define

$$\lambda := \frac{1}{2}\left[\frac{r-\bar{r}}{(2v_F\Delta)^2}\right] = -\frac{N''(0)}{24}$$

which we probationally take to be positive and note that is independent of Δ. Then we get after some manipulation

$$\mathcal{L}_{\text{eff}}^{(4)} = \frac{3\lambda}{4}(\Gamma_{\mu i}\Gamma_{\mu i})^2 - \frac{\lambda}{4}\left(\epsilon_{\mu\nu}\epsilon_{ij}\Gamma_{\mu i}\Gamma_{\nu j}\right)^2$$

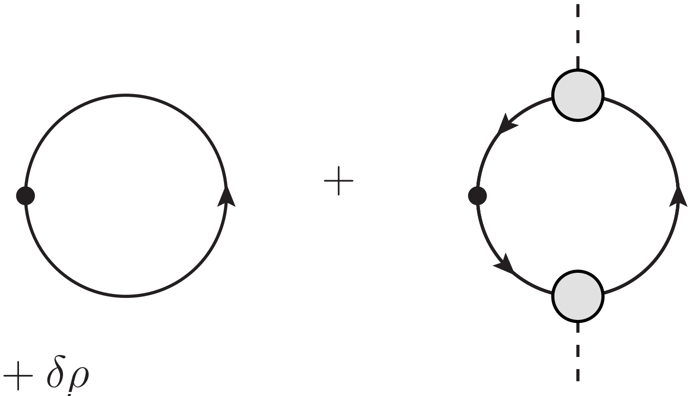
where we have expanded $(\Gamma_{\mu 1}\Gamma_{\mu 2})^2 = (\Gamma_{\mu 1}\Gamma_{\mu 1})(\Gamma_{\nu 2}\Gamma_{\nu 2}) - (\epsilon_{\mu\nu}\Gamma_{\mu 1}\Gamma_{\nu 2})^2$ and collected some common terms. These quartic terms also agrees with expectations based on the O(2)×O(2) symmetry analysis, as discussed in Appendix A.5.1. Note that the symmetry of the microscopic theory is $\mathbb{Z}_2 \times \mathrm{U}(1) \times \mathrm{O}(2) \cong \mathrm{O}(2) \times \mathrm{O}(2)$. The $\mathbb{Z}_2$ is time-reversal implemented by $\mathbf{k} \to -\mathbf{k}$ and complex conjugation. The U(1) factor is the relative phase between bands, generated globally by σ^z. The O(2) factor is the spatial rotational and mirror (parity) symmetry.

As it stands the β TRS broken phase is favored by this total term. In the next section we discuss the possibility of stabilizing an α phase by imposing a fixed density constraint which introduces a counter-term at the quartic order in Γ.

4.5.3 Fixed Density

The field $\Gamma_{\mu i}(q)$ is like a density wave for $q \neq 0$ with (orbital) quadrupolar character. Having a non-zero $\langle\Gamma(q=0)\rangle$ will lead to distortions of the Fermi-surfaces and changes to the density which we will try to account for here. Now the density diagram with $O(\Gamma^2)$ corrections is as shown below:

$$n(q=0) = \langle 0|T\psi^\dagger\psi|0\rangle =$$



$$= \rho + \delta\rho$$

Figure 4.4: The density ρ and its perturbative correction $\delta\rho$ due to Γ at order $O(\Gamma^2)$.

Here the "bare" density is given by the standard bubble as

$$\rho = \sum_m \int \mathrm{d}\xi \; N(\xi)\frac{1}{\beta}\sum_{k_0}\left(\frac{1}{ik_0 - \xi_{m\mathbf{k}}}\right) = \sum_m \int \mathrm{d}\xi \; N(\xi)n_F(\xi) = \rho_0 + (2v_F\Delta)r$$

here $\rho_0 := 2\int_{-\infty}^{0} \mathrm{d}\xi \; N(\xi)$ is the reference density when $\Delta = 0$ with zero splitting. Now the additional $O(\Gamma^2)$ contribution $\delta\rho(\Gamma)$ is given as below

$$\delta\rho(\Gamma) = \frac{N'(0)}{2}(\Gamma_{\mu i}\Gamma_{\mu i})$$

so

$$n(q=0) = \rho_0 + (2v_F\Delta)r + \frac{N'(0)}{2}(\Gamma_{\mu i}\Gamma_{\mu i}).$$

Then parametrize $\rho_0(\delta\mu) := 2\int_{-\infty}^{\delta\mu} \mathrm{d}\xi \; N(\xi)$ as a change in "bare" density due to a small change in chemical potential $\delta\mu$. Essentially $\delta\mu\psi_m^\dagger\psi_m$ will act as a counter-term to fixed the density by canceling $\delta\rho(\Gamma)$. Thus

$$\rho_0'(\delta\mu) = 2N(\delta\mu) \quad \Rightarrow \quad \rho_0(\delta\mu) \approx \rho_0 + 2N(0)\delta\mu$$

to lowest order. Demanding that $n(q=0)$ remain constant at its $\Gamma = 0$ value, we have to leading order the constraint equation

$$\rho_0(\delta\mu) + (2v_F\Delta)r(\delta\mu) + \frac{N'(+\delta\mu)}{2}(\Gamma_{\mu i}\Gamma_{\mu i}) = \rho_0 + (2v_F\Delta)r.$$

To leading order in smallness in $\Gamma_{\mu i}\Gamma_{\mu i}$ we can ignore the variation in the coefficient in $\Gamma_{\mu i}\Gamma_{\mu i}$. But

$$r(\delta\mu) = \frac{1}{2v_F\Delta}\int_{-v_F\Delta}^{v_F\Delta} \mathrm{d}\xi \; N(\xi - \delta\mu) = r - N'(0)\delta\mu.$$

Then we have to $O(\delta\mu)$ from solving the constraint equation

$$\delta\mu = -\frac{N'(0)}{2\left[(2v_F\Delta)N'(0) + 2N(0)\right]}\Gamma_{\mu i}\Gamma_{\mu i}.$$

Now this correction in the chemical potential leads to a correction in r as

$$\delta r = +\delta\mu N'(0) = -\frac{1}{2}\left[\frac{N'(0)^2}{2N(0) - (2v_F\Delta)N'(0)}\right](\Gamma_{\mu i}\Gamma_{\mu i})$$

and so the effective action acquires an additional "counter-term" arising from the quadratic term $\mathcal{L}^{(2)}_{\text{eff}} = -(r/2)\Gamma_{\mu i}\Gamma_{\mu i}$ as

$$-\frac{\delta r(\Gamma)}{2}(\Gamma_{\mu i}\Gamma_{\mu i}) = +\frac{1}{4}\left[\frac{N'(0)^2}{2N(0) - (2v_F\Delta)N'(0)}\right](\Gamma_{\mu i}\Gamma_{\mu i})^2 \approx +\frac{1}{8}\left(\frac{N'(0)^2}{N(0)}\right)(\Gamma_{\mu i}\Gamma_{\mu i})^2$$

in the approximation that $\left|\frac{N(0)}{N'(0)}\right| \gg 2v_F\Delta$, that is in the large DOS limit. Also note that this correction is regular in the $\Delta = 0$ limit. Hence the final counter-term that is added to the effective Lagrangian from fixing the density is

$$\delta\mathcal{L}^{(4)}_{\text{eff}} = +\left(\frac{N'(0)^2}{8N(0)}\right)(\Gamma_{\mu i}\Gamma_{\mu i})^2 \tag{4.38}$$

which leads to the total quartic term

$$\mathcal{L}_{\text{eff}}^{(4)} = \left(\frac{N'(0)^2}{8N(0)} - \frac{N''(0)}{32}\right)(\Gamma_{\mu i}\Gamma_{\mu i})^2 + \left(\frac{N''(0)}{96}\right)(\epsilon_{\mu\nu}\epsilon_{ij}\Gamma_{\mu i}\Gamma_{\nu j})^2 \tag{4.39}$$

$$= \left(\frac{\alpha'}{4} + \frac{3\lambda}{4}\right)(\Gamma_{\mu i}\Gamma_{\mu i})^2 - \frac{\lambda}{4}(\epsilon_{\mu\nu}\epsilon_{ij}\Gamma_{\mu i}\Gamma_{\nu j})^2 \tag{4.40}$$

where $\alpha' := \frac{N'(0)^2}{2N(0)}$ is newly defined parameter. Hence we can stabilize an α phase when we have $N''(0) > 0$ but the require that

$$\frac{N'(0)^2}{N(0)} > \frac{N''(0)}{4} > 0.$$

4.5.4 Final Effective Action

First, we focus on just the uniform-static limit $q = 0$ when the effective action S_{eff} becomes a free-energy functional. Putting the quadratic and quartic interaction contributions together, we arrive at

$$\mathcal{F}_{\text{eff}} = \left(\frac{\alpha'}{4} + \frac{3\lambda}{4}\right)(\Gamma_{\mu i}\Gamma_{\mu i})^2 - \frac{\lambda}{4}(\epsilon_{\mu\nu}\epsilon\Gamma_{\mu i}\Gamma_{\nu j})^2 + \frac{1}{2}\left(\frac{1}{|f(0)|} - r\right)\Gamma_{\mu i}\Gamma_{\mu i}. \tag{4.41}$$

Next the lowest order gradient terms are

$$\mathcal{L}_{\text{eff}}^{\text{grad}}(q) = \frac{\kappa|\mathbf{q}|^2}{2}\Gamma_{\mu i}(q)\Gamma_{\mu i}(-q) + \frac{q_0}{2}\left(\frac{r}{2v_F\Delta}\right)\epsilon_{\mu\nu}\Gamma_{\mu i}(q)\Gamma_{\nu i}(-q)$$

which in real-space is

$$\mathcal{L}_{\text{eff}}^{\text{grad}} = \frac{\kappa}{2}\,\partial_a\Gamma_{\mu i}(\mathbf{x},\tau)\partial_a\Gamma_{\mu i}(\mathbf{x},\tau) - \frac{i}{2}\left(\frac{r}{2v_F\Delta}\right)\epsilon_{\mu\nu}\epsilon_{ij}\Gamma_{\mu i}(\mathbf{x},\tau)[\partial_\tau\Gamma_{\nu j}(\mathbf{x},\tau)].$$

Combining everything and defining some new parameters, we have the final effective action as

$$\mathcal{L}_{\text{eff}} = \frac{\alpha}{4}(\Gamma_{\mu i}\Gamma_{\mu i})^2 - \frac{\lambda}{4}(\epsilon_{\mu\nu}\epsilon\Gamma_{\mu i}\Gamma_{\nu j})^2 + \frac{\rho}{2}(\Gamma_{\mu i}\Gamma_{\mu i}) + \frac{\kappa}{2}\,\partial_a\Gamma_{\mu i}\partial_a\Gamma_{\mu i} - \frac{i\beta}{2}\epsilon_{\mu\nu}\epsilon_{ij}\Gamma_{\mu i}\partial_\tau\Gamma_{\nu j} \quad (4.42)$$

with

$$\begin{aligned} r &= N(0) + \frac{(v_F\Delta)^2}{6}N''(0), & \lambda &= -\frac{N''(0)}{24}, & \rho &= \frac{1}{|f(0)|} - r, \\ \alpha' &= \frac{N'(0)^2}{2N(0)}, & \alpha &= \alpha' - 3\lambda = \frac{N'(0)^2}{2N(0)} - \frac{N''(0)}{8}, & \beta &= \frac{r}{2v_F\Delta} \end{aligned}$$

The bilinear cross-product at lowest order $\propto \epsilon_{\mu\nu}\vec{\Gamma}_{\mu i}(\partial_\tau\vec{\Gamma}_{\nu i})$ is recognized to be a Berry phase term and had been previously encountered in the quadratic band touching[You and Fradkin, 2013] model and fractional Quantum Hall liquid[Maciejko et al., 2013] versions of this nematic driven QAH phase. In real time this term takes the form

$$\mathcal{L}_{\text{Berry}} = -\frac{\beta}{2}\epsilon_{\mu\nu}\Gamma_{\mu i}(\partial_t\Gamma_{\nu i}). \quad (4.43)$$

which because the real $\Gamma_{\mu i}$ transforms as a reflection in O(2) space under time-reversal, means that this term is actually time-reversal invariant, despite appearances. Also the presence of this term leads to a dynamical exponent of $z = 2$ instead of $z = 3$ encountered in the non-split Fermi liquid phase[Wu et al., 2007], $\Delta = 0$. Power counting of momenta naturally gives scaling dimensions, $[\Gamma] = +1$ for the bosonic field, $[\rho] = 2$ relevant and $[\alpha] = [\lambda] = 0$ marginal. Thus per conventional Landau theory, the critical regime occurs towards a Pomenranchuk instability in the vicinity where $|f(0)|r = 1$, with the spontaneous symmetry breaking stabilized by the quartic interaction. This picture is at least qualitatively accurate near the critical point.

Next going over to real-time, the effective Lagrangian takes the form

$$\mathcal{L}_{\text{eff}} = -\frac{\beta}{2}\epsilon_{\mu\nu}\Gamma_{\mu i}(\partial_t\Gamma_{\nu i}) + \frac{\kappa}{2}\left(\nabla\Gamma_{\mu i}\right)\cdot\left(\nabla\Gamma_{\mu i}\right) + \frac{\rho}{2}\Gamma_{\mu i}\Gamma_{\mu i} + \frac{\alpha}{4}\left(\Gamma_{\mu i}\Gamma_{\mu i}\right)^2 - \frac{\lambda}{4}\left(\epsilon_{\mu\nu}\epsilon_{ij}\Gamma_{\mu i}\Gamma_{\nu j}\right)^2 \tag{4.44}$$

Next formally de-quantizing back to a classical limit, we recognize that the Γ_{xi} and Γ_{yi} are conjugate fields in the Hamiltonian sense. The Berry phase term induces the following classical (equal time) Poisson bracket

$$\{\Gamma_{\mu i}(\mathbf{x}), \Gamma_{\nu j}(\mathbf{y})\} = -\left(\frac{2}{\beta}\right)\epsilon_{\mu\nu}\delta_{ij}\;\delta^2(\mathbf{x}-\mathbf{y}). \tag{4.45}$$

This can also be appreciated from looking at the equations of motion implied by the effective action. Variation of $\delta\Gamma$ yields

$$\beta\,\partial_t\Gamma_{\nu i} = \left[\rho + \alpha\left(\Gamma_{\mu' i'}\Gamma_{\mu' i'}\right)\right]\epsilon_{\mu\nu}\Gamma_{\mu i} + \lambda\left(\epsilon_{\mu'\nu'}\epsilon_{i'j'}\Gamma_{\mu' i'}\Gamma_{\nu' j'}\right)\epsilon_{ij}\,\Gamma_{\nu j} - \kappa\,\epsilon_{\mu\nu}\,\nabla^2\Gamma_{\mu i} \tag{4.46}$$

which looks like a non-linear Landau-Lifshitz-Gilbert equation with a magnetic field in the $+\sigma^z$ direction.

4.6 Electromagnetic Response

4.6.1 Response diagrams and Ward identities

We consider only, the response in the limit where $\Gamma_{\mu i}$ is slow compared to the probing EM field $A_\mu(x)$ such that we can neglect couplings to gradients $\partial_x\Gamma_{\mu i}(x)$. In this limit $\Gamma_{\mu i}$ acts like a background field for the ψ fermions from which response is computed. The Lagrangian minimally coupled to EM with a background Γ is

$$L[\Gamma] = \int \mathrm{d}^2x \left[\psi^\dagger D_\tau \psi + \psi^\dagger [(\xi(-i\mathbf{D})) + v_F \Delta\sigma^z]\psi - \Gamma_{\mu i}\ \psi^\dagger \left(\frac{\sigma^\mu \tau^i_{ab}}{k_F^2}\right) D_a D_b \psi \right]$$

where

$$D_a = \frac{\partial}{\partial x_a} + ieA_a, \qquad D_\tau = \frac{\partial}{\partial \tau} - eA_0$$

and where the covariant derivative operators are understood to act symmetrically on the left and right. That is $D \equiv 1/2(\overrightarrow{D} - \overleftarrow{D})$ etc. The full minimally coupled single particle Green's function in the presence of A and Γ background fields is

$$\begin{aligned}
-\hat{\mathcal{G}} =& \hat{D}_\tau + \xi(-i\hat{\mathbf{D}}) + v_F\Delta\sigma^z + \Gamma_{\mu i}\left(\frac{\sigma^\mu \tau^i_{ab}}{k_F^2}\right)(-i\hat{D}_a)(-i\hat{D}_b) \\
=& -i\hat{k}_0 + \xi(\hat{\mathbf{k}}) + v_F\Delta\sigma^z + \Gamma_{\mu i}\left(\frac{\sigma^\mu \tau^i_{ab}}{k_F^2}\right)\hat{k}_a\hat{k}_b \\
&- eA_0(\hat{\mathbf{x}}) + \tfrac{1}{2}\{\partial_a \xi(\hat{\mathbf{k}}), eA_a(\hat{\mathbf{x}})\} + \frac{1}{2\cdot 3!}\left(e^2\partial^2_{ab}\xi(\hat{\mathbf{k}})A_a(\hat{\mathbf{x}})A_b(\hat{\mathbf{x}}) + \text{cyclic}\right) \\
&+ \frac{1}{2!}\left(2e\Gamma_{\mu i}\left(\frac{\sigma^\mu \tau^i_{ab}}{k_F^2}\right) +\text{cyclic}\right) + e^2\Gamma_{\mu i}\left(\frac{\sigma^\mu \tau^i_{ab}}{k_F^2}\right)A_a(\hat{\mathbf{x}})A_b(\hat{\mathbf{x}})
\end{aligned}$$

where the hats are meant to emphasize the operator character of the various terms, and we have expanded only to the linear response limit $O(A^2)$. The cyclic permutations ensure that there is not ambiguity in the operator ordering. Essentially what has been done is the Peierls substitution and a semi-classical substitution of the dispersion up to second oder in $A-$fields. See Blount [Blount, 1962] for the best discussion.

The current density operators are derived from A field variations of the action. Equivalently the (1st quantized) covariant ($e\times$) velocity operators[1] are derived from A derivatives

[1] The apparent incorrect sign in the velocity of current operator has to do with the Euclidean signature of imaginary time. Transforming back to real-time fixes the sign.

of the inverse propagator

$$\hat{v}^a[A] = -\frac{\partial \hat{\mathcal{G}}^{-1}}{\partial A_a} = e\partial_a \xi(\hat{\mathbf{k}}) + \frac{e^2}{2}\{\partial^2_{ab}\xi(\hat{\mathbf{k}}), A_b(\hat{\mathbf{x}})\} + 2e\Gamma_{\mu i}\left(\frac{\sigma^\mu \tau^i_{ab}}{k_F^2}\right)\hat{k}_b + 2e^2\left(\frac{\sigma^\mu \tau^i_{ab}}{k_F^2}\right)A_b(\hat{\mathbf{x}})$$
$$\hat{v}^0[A] = -\frac{\partial \hat{\mathcal{G}}^{-1}}{\partial A_0} = -e \quad \leftarrow \text{charge operator}$$

The physical velocity operator is linear in $A_a(\hat{\mathbf{x}})$ and is gauge covariant. The additional terms linear in A, may be thought of as diamagnetic contributions. The operators more conventionally known as the velocity operator is obtained by setting $A = 0$, which still is Γ dependent, but diagonal in $\mathbf{k}$. Finally the local physical currents are

$$j^a[A](x) = \psi^\dagger(x)\hat{v}^a[A]\psi(x) \tag{4.47}$$

$$j^0[A](x) = -e\psi^\dagger(x)\psi(x). \tag{4.48}$$

Next we need expressions for the linear response current in the presence of the background fields.

$$\langle j^\mu[A]\rangle_{A,\Gamma} = Z^{-1}\int \mathrm{d}[\psi^\dagger,\psi]\; j^\mu[A]\; \mathrm{e}^{-S[\psi^\dagger,\psi,\Gamma,A]}$$

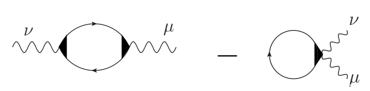
Then expanding the action to 2nd order in fields A yields

$$\begin{aligned} S[\psi^\dagger,\psi,\Gamma,A] &= S[\psi^\dagger,\psi,\Gamma,0] + A_\nu \frac{\delta S[\psi^\dagger,\psi,\Gamma,0]}{\delta A_\nu} + \frac{1}{2!}A_\nu\left(\frac{\delta^2 S[\psi^\dagger,\psi,\Gamma,0]}{\delta A_\nu \delta A_\mu}\right)A_\mu \\ &= S[\psi^\dagger,\psi,\Gamma,0] + A_\nu j^\nu[A] \end{aligned}$$

here $j^b[A]$ is linear in A which included the diamagnetic contribution. Also the multiplications are understood to involve inner-products with the space-time integral.

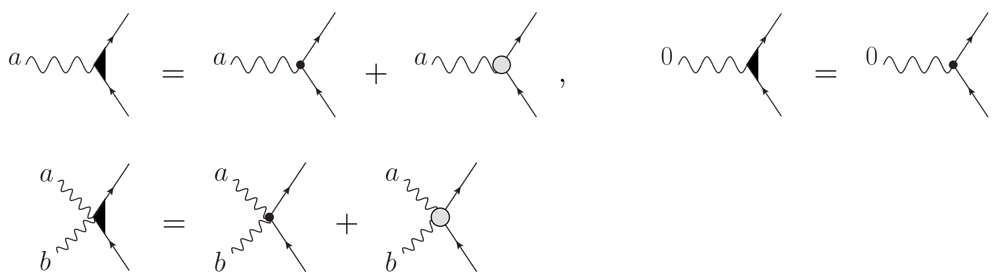
Then to linear response order

$$
\begin{aligned}
\langle j^\mu[A]\rangle_{A,\Gamma} &= Z^{-1}\int \mathrm{d}[\psi^\dagger,\psi]\;(+j^\mu[A])(1-A_\nu j^\nu[A])\;\mathrm{e}^{-S[\psi^\dagger,\psi,\Gamma,0]} \\
&= -A_\nu Z^{-1}\int \mathrm{d}[\psi^\dagger,\psi]\left(j^\mu[0]j^\nu[0]-\frac{\delta j^\mu[0]}{\delta A_\nu}\right)\mathrm{e}^{-S[\psi^\dagger,\psi,\Gamma,0]} \\
&= +A_\nu\Bigg(
\end{aligned}
$$



$$\Bigg)$$

where we have only retained terms linear in the probe field A producing current j^μ. Here the lines are Green's functions computed in the presence of Γ and the current vertices are $j^\mu[0]$. Each gauge invariant current insertion gives its own conserved (transverse) set of polarization tensors once tadpole contributions are included. The vertices $j^\mu[0]$, $\frac{\delta j^\mu[0]}{\delta A_\nu}$ have the following diagrammatic representations

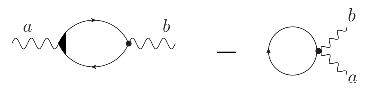


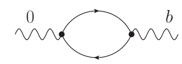
where we have the contraction with $\Gamma_{\mu i}$ is implied in the (●) vertices. We can then naturally split the total response current into separate gauge invariant contributions

$$
\begin{aligned}
j^a[A] =& e\psi^\dagger(x)\partial_a\xi\psi(x) + e^2\left[\psi^\dagger(x)\partial^2_{ab}\xi\psi(x)\right]A_b(x) + \frac{2e}{k_F^2}\Gamma_{\mu i}\tau^i_{ab}\psi^\dagger(x)\sigma^\mu k_b\psi(x) \\
&+ \frac{2e^2}{k_F^2}\Gamma_{\mu i}\tau^i_{ab}\psi^\dagger(x)\sigma^\mu\psi(x)A_b(x) \\
=& j^a_1[A] + j^a_2[\Gamma,A] \\
j^a_1[A] =& e\psi^\dagger(x)\partial_a\xi\psi(x) + e^2\left[\psi^\dagger(x)\partial^2_{ab}\xi\psi(x)\right]A_b(x) \\
j^a_2[\Gamma,A] =& \frac{2e}{k_F^2}\Gamma_{\mu i}\tau^i_{ab}\psi^\dagger(x)\sigma^\mu k_b\psi(x) + \frac{2e^2}{k_F^2}\Gamma_{\mu i}\tau^i_{ab}\psi^\dagger(x)\sigma^\mu\psi(x)A_b(x)
\end{aligned}
$$

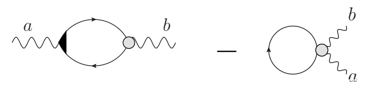
Here j_2^a explicitly depends on $\Gamma_{\mu i}$. Note that both current density fields are separately gauge invariant. Well actually j_1^a is only gauge invariant up to linear order in $O(k+eA)$ which is typical of linear response calculations. j^0 being the physical charge is always gauge invariant.

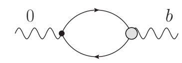
Diagrammatically the linear response gauge invariant currents are

$\langle j_1^b[A]\rangle = A_a \Big($ 

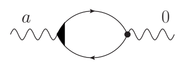
$\Big) + A_0 \Big($ 

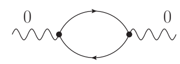
$\Big)$

$\langle J_2^b[A]\rangle = A_a \Big($ 

$\Big) + A_0 \Big($ 

$\Big)$

$\langle j^0\rangle = A_a \Big($ 

$\Big) + A_0 \Big($ 

$\Big)$ (4.49)

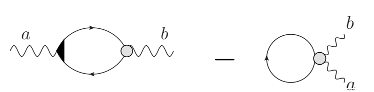
To appreciate the gauge invariance of each set, consider $j_2[A]$

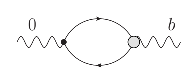
$$\langle j_2^b[A]\rangle = Z^{-1}\int \mathrm{d}[\psi^\dagger,\psi]\ \left(\frac{2e}{k_F^2}\right)\tau^i_{ab}\ [\psi\sigma^\mu(k+eA)_b\psi]\ \mathrm{e}^{-S[\psi^\dagger,\psi,\Gamma,A]}$$

then applying a gauge transform $A \to A + d\varphi$, $\psi \to \psi \mathrm{e}^{-i\varphi}$, we see that the correlator remains unchanged because the action and the current density are gauge invariant by construction. Thus

$$\langle j_2^b[A]\rangle = \langle j_2^b[A+d\varphi]\rangle = \langle j_2^b[A]\rangle = \int\ \partial_\mu\varphi\frac{\delta}{\delta A_\mu}\langle j_2^b[A]\rangle$$

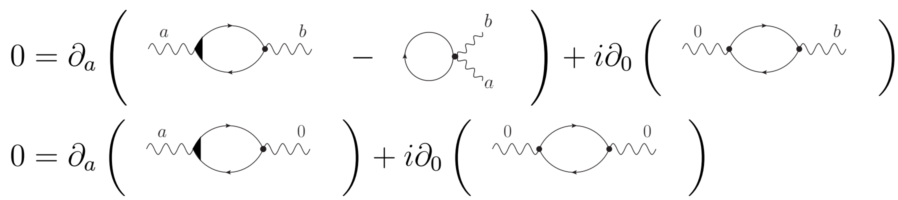
$$\Rightarrow 0 = \partial_\mu\left(\frac{\delta}{\delta A_\mu}\langle j_2^b[A]\rangle\right)$$

$\Rightarrow 0 = \partial_a \Big($ 

$\Big) + i\partial_0 \Big($ 

$\Big)$

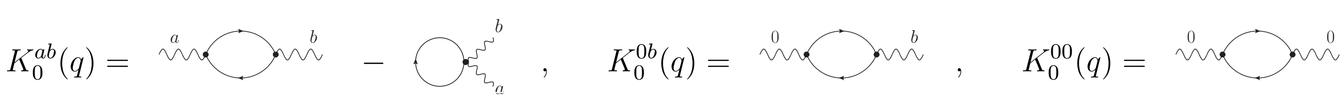
from integrating by parts, then expanding to linear order in A. The $(-i)$ factor in the temporal derivative has to do with the imaginary-time signature. A similar argument gives the following Ward identities as well.



Note that the fermion lines here are dressed with Γ corrections. And so these diagrams then have to be separately expanded to $O(\Gamma^2)$ which is the order in which we are interested in for the effective action. Defining respectively the following linear response kernels (q is the incoming momentum),

$$
\begin{aligned}
K_0^{\mu b}(q) &= \left.\frac{\delta\langle j_1^b[A]\rangle_{\Gamma=0,A}}{\delta A_\mu(q)}\right|_{A=0} \\
K_0^{\mu 0}(q) &= \left.\frac{\delta\langle j^0\rangle_{\Gamma=0,A}}{\delta A_\mu(q)}\right|_{A=0} \\
K_1^{\mu b}(q) &= \left.\frac{\delta\langle j_1^b[A]\rangle_{\Gamma,A}}{\delta A_\mu(q)}\right|_{A=0} - K_0^{\mu b}(q) \\
K_1^{\mu 0}(q) &= \left.\frac{\delta\langle j^0\rangle_{\Gamma,A}}{\delta A_\mu(q)}\right|_{A=0} - K_0^{\mu 0} q \\
K_2^{\mu b}(q) &= \left.\frac{\delta\langle j_2^b[\Gamma,A]\rangle_{\Gamma,A}}{\delta A_\mu(q)}\right|_{A=0}
\end{aligned}
$$

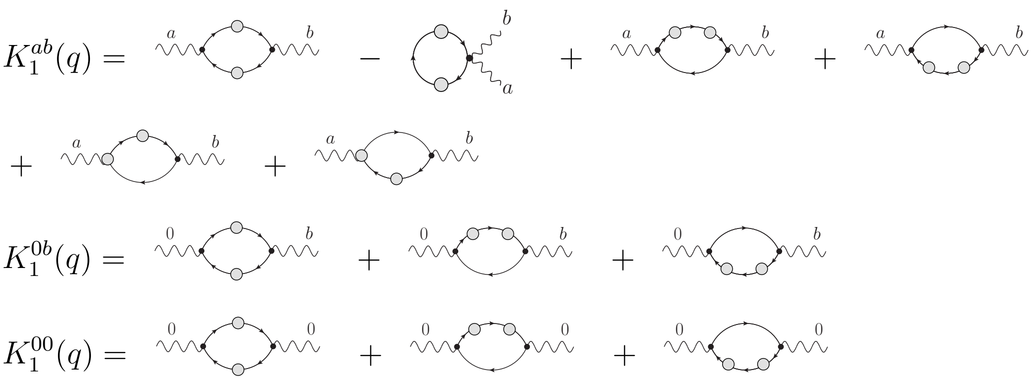
which are individually transverse. The first and second kernels are the conventional linear response when $\Gamma = 0$ and are transverse by themselves, which makes the K_1 pair mutually transverse. We then expand each response kernel to $O(\Gamma^2)$



(4.50)

which are the conventional response kernels, which are followed by

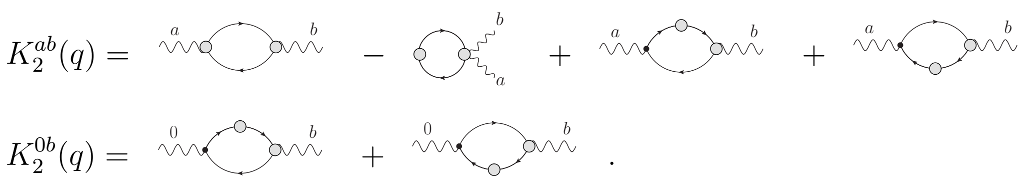
$K_1^{ab}(q) =$, $K_1^{0b}(q) =$, $K_1^{00}(q) =$



(4.51)

which are diagrams dressed by $O(\Gamma^2)$ corrections. The remaining set is

$K_2^{ab}(q) =$, $K_2^{0b}(q) =$



(4.52)

The kernels K_1^{b0} are similarly defined to K_1^{0b}, and K_2 does not have a density-density response counterpart because j_2 does not coupled to A_0. These kernels being all transverse means that they satisfy the following Ward identities

$$q_a K_m^{ab}(q) + iq_0 K_m^{0b}(q) = 0 \tag{4.53}$$

$$q_a K_m^{a0}(q) + iq_0 K_m^{00}(q) = 0 \tag{4.54}$$

where $m = 0, 1, 2$ where defined.

For convenience we list here some definitions for parameters and notation used in expression the response kernels.

$$\xi_1(\mathbf{k}) := \xi_{\mathbf{k}} + v_F\Delta, \qquad \xi_2(\mathbf{k}) := \xi_{\mathbf{k}} - v_F\Delta,$$

$$v_F := \left.\frac{\partial \xi_{\mathbf{k}}}{\partial |\mathbf{k}|}\right|_{\xi_{\mathbf{k}}=0}$$

$$r := \frac{1}{2v_F\Delta}\int_{-v_F\Delta}^{+v_F\Delta} N(\xi)\mathrm{d}\xi, \qquad \bar{r} := \frac{N(+v_F\Delta) + N(-v_F\Delta)}{2},$$

$$s := \frac{N(+v_F\Delta) - N(-v_F\Delta)}{2v_F\Delta}$$

$$\|q\| := \sqrt{v_F|\mathbf{q}| + |q_0|} \qquad G_{1,2} := \frac{1}{iq_0 - \xi_{1,2}(\mathbf{k})}$$

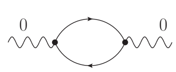
$$v_F\delta q := v_F\mathbf{q}\cdot\hat{\mathbf{k}} \equiv v_F|\mathbf{q}|\cos(\theta_{\mathbf{k}} - \theta_{\mathbf{q}})$$

and the quadrupole tensors $\tau^1 \equiv \sigma^z, \tau^2 \equiv \sigma^x$ such that $\hat{\mathbf{k}}^T\tau^i\hat{\mathbf{k}} = (\cos 2\theta_k, \sin 2\theta_k)_i$.

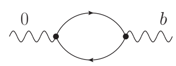
4.6.2 Conventional Polarization Diagrams

These diagrams do not couple $\Gamma_{\mu i}$ and represent the "bare" electromagnetic response. In deriving these expressions, we used the gradient expansion simplifications and angular expansions detailed in Appendices A.4 and A.2.

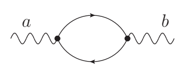
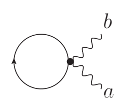
$K_0^{00}(q) =$



$$= -e^2\int\frac{\mathrm{d}^2k}{(2\pi)^2}\,\frac{1}{\beta}\sum_{k_0}\left(\frac{v_F\delta q}{v_F\delta q - iq_0}\right)(G_1^2 + G_2^2)$$
$$= 2e^2\bar{r}\left(1 - \frac{|q_0|}{\|q\|}\right)$$

$$K_0^{0b}(q) = $$  $$= e^2 \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \, \frac{1}{\beta} \sum_{k_0} \partial_b \xi_{\mathbf{k}} \left(\frac{v_F \delta q}{v_F \delta q - i q_0} \right) (G_1^2 + G_2^2)$$

$$= -2ie^2 \bar{r} v_F \frac{|q_0|}{\|q\|} \left(\frac{v_F |\mathbf{q}|}{\|q\| + |q_0|} \right) \begin{pmatrix} \cos\theta_q \\ \sin\theta_q \end{pmatrix}_b$$

$$K_0^{ab}(q) = \Bigg($$  $$-$$  $$\Bigg)$$

$$= -e^2 \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \, \frac{1}{\beta} \sum_{k_0} \partial_a \xi_{\mathbf{k}} \partial_b \xi_{\mathbf{k}} \left(\frac{i q_0}{v_F \delta q - i q_0} \right) (G_1^2 + G_2^2)$$

$$= e^2 \bar{r} v_F^2 \frac{|q_0|}{\|q\|} \left[-\delta_{ab} + \left(\frac{v_F |\mathbf{q}|}{\|q\| + |q_0|} \right)^2 \begin{pmatrix} \cos 2\theta_q & \sin 2\theta_q \\ \sin 2\theta_q & -\cos 2\theta_q \end{pmatrix}_{ab} \right]$$

where $(\cos\theta_q, \sin\theta_q)^T = \hat{\mathbf{q}}$ and $\begin{pmatrix} \cos 2\theta_q & \sin 2\theta_q \\ \sin 2\theta_q & -\cos 2\theta_q \end{pmatrix}_{ab} = 2\hat{q}_a \hat{q}_b - \delta_{ab}.$

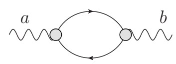
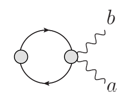
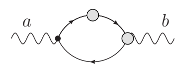
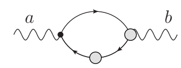
Also $K_0^{b0}(q) = K_0^{0b}(q)$.

The Ward identity $q_a K_0^{ab}(q) + i q_0 K_0^{0b}(q) = 0$ can be easily checked in the integrated and un-integrated forms. Coupling to the Γ nematic field will introduce more complicated harmonics in the EM response kernels as well as a Hall conductivity (Chern-Simons) contribution. Also when analytically continuing to the retarded branch, we use the following

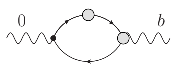
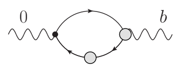
$$|q_0| \to -i\omega + 0^+, \qquad \|q\| = \begin{cases} \sqrt{v_F^2 |\mathbf{q}|^2 - \omega^2}, & \left(\frac{\omega}{v_F |\mathbf{q}|}\right)^2 > 1 \\ -i\sqrt{\omega^2 - v_F^2 |\mathbf{q}|^2}, & \left(\frac{\omega}{v_F |\mathbf{q}|}\right)^2 < 1 \end{cases}$$

4.6.3 Response of $K_2^{\mu b}$ and its Ward identity

These are diagrams which have an outgoing (to the right) current vertex that originates from the quadrupole vertex (shaded blobs). They produce the first non-trivial Hall response.

$K_2^{ab}(iq_0, \mathbf{q}) = \Bigg($  $-$  $+$  $+$  $\Bigg)$

$$
\begin{aligned}
&= -\left(\frac{4e^2}{k_F^4}\right)\int \frac{\mathrm{d}^2k}{(2\pi)^2}\,\frac{1}{\beta}\sum_{k_0}\Bigg[+2iq_0(\tau^i\mathbf{k})_a(\tau^j\mathbf{k})_b\left(\frac{G_1-G_2}{(2v_F\Delta)^2}\right)i\epsilon_{\mu\nu} \\
&+\left(\frac{iq_0}{v_F\delta q - iq_0}\right)\partial_a\xi_{\mathbf{k}}\;(\mathbf{k}^T\tau^i\mathbf{k})(\tau^j\mathbf{k})_b\left(\frac{G_1^2-G_2^2}{2v_F\Delta}\right)\delta_{\mu\nu} \\
&-\left(\frac{iq_0}{v_F\delta q - iq_0}\right)\partial_a\xi_{\mathbf{k}}\;(\mathbf{q}^T\tau^i\mathbf{k})(\tau^j\mathbf{k})_b\left(\frac{G_1^2+G_2^2}{2v_F\Delta}\right)i\epsilon_{\mu\nu}\Bigg]\Gamma_{\mu i}\Gamma_{\nu j} \\
&= -\left(\frac{4e^2}{k_F^2}\right)\left(\frac{r}{2v_F\Delta}\right)(\epsilon_{ij}\epsilon_{\mu\nu}\Gamma_{\mu i}\Gamma_{\nu j})\;q_0\epsilon_{ab} \\
&+\left(\frac{2e^2}{k_F^2}\right)\left(\frac{\bar{r}}{2v_F\Delta}\right)(\epsilon_{ij}\epsilon_{\mu\nu}\Gamma_{\mu i}\Gamma_{\nu j})q_0\left(1-\frac{|q_0|}{\|q\|}\right)\left\{\epsilon_{ab}+\begin{pmatrix}-\sin 2\theta_q & \cos 2\theta_q \\ \cos 2\theta_q & \sin 2\theta_q\end{pmatrix}_{ab}\right\} \\
&+\left(\frac{e^2 s}{k_F}\right)\Gamma_{\mu i}\Gamma_{\mu j}\left(\frac{v_F|\mathbf{q}|}{\|q\|}\right) \\
&\quad\times\Bigg\{+\delta_{ij}\delta_{ab} \\
&-\left(\frac{v_F|\mathbf{q}|}{\|q\|+|q_0|}\right)^2\Bigg[\delta_{ij}\begin{pmatrix}\cos 2\theta_q & \sin 2\theta_q \\ \sin 2\theta_q & -\cos 2\theta_q\end{pmatrix}_{ab}+\sigma^z_{ij}\begin{pmatrix}\cos 2\theta_q & -\sin 2\theta_q \\ -\sin 2\theta_q & -\cos 2\theta_q\end{pmatrix}_{ab} \\
&+\sigma^x_{ij}\begin{pmatrix}\sin 2\theta_q & \cos 2\theta_q \\ \cos 2\theta_q & -\sin 2\theta_q\end{pmatrix}_{ab}\Bigg] \\
&-\left(\frac{v_F|\mathbf{q}|}{\|q\|+|q_0|}\right)^4\Bigg[\sigma^z_{ij}\begin{pmatrix}\cos 4\theta_q & -\sin 4\theta_q \\ \sin 4\theta_q & \cos 4\theta_q\end{pmatrix}_{ab}+\sigma^x_{ij}\begin{pmatrix}\sin 4\theta_q & \cos 4\theta_q \\ -\cos 4\theta_q & \sin 4\theta_q\end{pmatrix}_{ab}\Bigg]\Bigg\}
\end{aligned}
$$

$$K_2^{0b}(iq_0,\mathbf{q}) = \Bigg($$  $$+$$  $$\Bigg)$$

$$
\begin{aligned}
&= \left(\frac{4e^2}{k_F^4}\right)\int \frac{\mathrm{d}^2k}{(2\pi)^2}\,\frac{1}{\beta}\sum_{k_0}\Bigg[\left(\frac{v_F\delta q}{v_F\delta q - iq_0}\right)(\mathbf{k}^T\tau^i\mathbf{k})(\tau^j\mathbf{k})_b\left(\frac{G_1^2-G_2^2}{2v_F\Delta}\right)\delta_{\mu\nu} \\
&\qquad +2(\mathbf{q}^T\tau^i\mathbf{k})(\tau^j\mathbf{k})_b\left(\frac{G_1-G_2}{(2v_F\Delta)^2}\right)i\epsilon_{\mu\nu} \\
&\qquad -\left(\frac{v_F\delta q}{v_F\delta q - iq_0}\right)(\mathbf{q}^T\tau^i\mathbf{k})(\tau^j\mathbf{k})_b\left(\frac{G_1^2+G_2^2}{2v_F\Delta}\right)i\epsilon_{\mu\nu}\Bigg]\Gamma_{\mu i}\Gamma_{\nu j} \\
&= -i\left(\frac{4e^2}{k_F^2}\right)\left(\frac{r}{2v_F\Delta}\right)(\epsilon_{ij}\epsilon_{\mu\nu}\Gamma_{\mu i}\Gamma_{\nu j})\;q_a\epsilon_{ab} \\
&\quad +i\left(\frac{4e^2}{k_F^2}\right)\left(\frac{\bar{r}}{2v_F\Delta}\right)(\epsilon_{ij}\epsilon_{\mu\nu}\Gamma_{\mu i}\Gamma_{\nu j})\left(1-\frac{|q_0|}{\|q\|}\right)q_a\epsilon_{ab} \\
&\quad +i\left(\frac{2e^2 s}{k_F}\right)(\Gamma_{\mu i}\Gamma_{\nu j})\frac{q_0}{\|q\|}\Bigg\{\left(\frac{v_F|\mathbf{q}|}{\|q\|+|q_0|}\right)\delta_{ij}\begin{pmatrix}\cos\theta_q\\ \sin\theta_q\end{pmatrix}_b \\
&\qquad +\left(\frac{v_F|\mathbf{q}|}{\|q\|+|q_0|}\right)^3\left[\sigma_{ij}^z\begin{pmatrix}-\cos 3\theta_q\\ \sin 3\theta_q\end{pmatrix}_b - \sigma_{ij}^x\begin{pmatrix}\sin 3\theta_q\\ \cos 3\theta_q\end{pmatrix}_b\right]\Bigg\}
\end{aligned}
$$

where a tedious calculation will show a line by line cancellation in the Ward identity $q_aK_2^{ab} + iq_0K_2^{0b} = 0$. Moreover the first lines of either K_2 contribute to a Hall (Chern-Simons) response. That is they contribute to the effective action as

$$\mathcal{L}_{\mathrm{CS}} = \left(\frac{4e^2}{k_F^2}\right)\left(\frac{r}{2v_F\Delta}\right)(\epsilon_{ij}\epsilon_{\mu\nu}\Gamma_{\mu i}\Gamma_{\nu j})\epsilon_{ab}\left[q_0A_a(q)A_b(-q) + iq_aA_0(q)A_b(-q)\right]$$

or in real-time with the substitutions $\tau \to it,\ \partial_\tau \to -i\partial_t,\ A_0 \to A_0$,

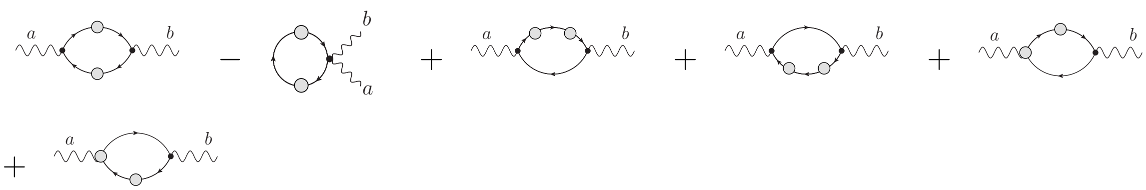
$$\mathcal{L}_{\mathrm{CS}} = \left(\frac{4e^2}{k_F^2}\right)\left(\frac{r}{2v_F\Delta}\right)(\epsilon_{ij}\epsilon_{\mu\nu}\Gamma_{\mu i}\Gamma_{\nu j})\epsilon_{ab}\left[A_a(x)\partial_tA_b(x) + A_0(x)\partial_aA_b(x)\right]$$

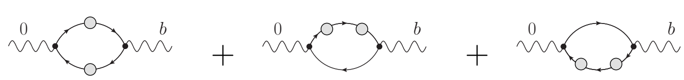
From the above polarization tensor, we can read off the DC-limit Hall conductance to be

$$\sigma_{ab} = \left(\frac{2e^2}{k_F^2}\right)\left(\frac{r}{v_F\Delta}\right)(\epsilon_{ij}\epsilon_{\mu\nu}\Gamma_{\mu i}\Gamma_{\nu j})\epsilon_{ab} \tag{4.55}$$

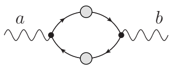
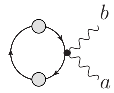
This conductance is unquantized and non-vanishing in a β_1 phase($\Gamma_{\mu i}$ are perpendicular to each other). On the other hand, it is equal to zero in the α_1 phase($\Gamma_{\mu i}$ are parallel to each other).

4.6.4 Response of $K_1^{\mu b}$ and its Ward identity

$K_1^{ab}(q) =$ 

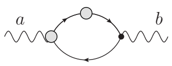
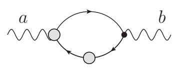
$K_1^{0b}(q) =$ 

Since this set only contributes to the longitudinal conductivity and it is also a correction we will not calculate the response explicitly but instead make use of the OPE-like expressions in momentum space to write them in terms of the integral and demonstrate that the Ward identity is satisfied. This set is very complicated so to get the most amount of cancellation as early as possible, we will look at the subset of the diagrams that should intutively cancelled amongst themselves.

( $-$ )

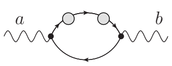
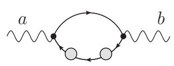
$$= \frac{2e^2}{k_F^4}\int \frac{d^2k}{(2\pi)^2}\frac{1}{\beta}\sum_{k_0}\{(\mathbf{k}^T\tau^i\mathbf{k})(\mathbf{k}^T\tau^j\mathbf{k})\partial_a\xi_{\mathbf{k}}\partial_b\xi_{\mathbf{k}}\left[G_1G_2(G_1^2+G_2^2)-\left(\frac{iq_0}{v_F\delta q-iq_0}\right)\frac{G_1^2+G_2^2}{(2v_F\Delta)^2}\right]$$

$$+2(\tau^i\mathbf{k})_a(\mathbf{k}^T\tau^j\mathbf{k})\partial_b\xi_{\mathbf{k}}\left[\frac{G_1^2-G_2^2}{2v_F\Delta}\right]\}\Gamma_{\mu i}\Gamma_{\mu j}$$

( $+$ )

$$= \frac{4e^2}{k_F^4}\int \frac{d^2k}{(2\pi)^2}\frac{1}{\beta}\sum_{k_0}\{(\tau^i\mathbf{k})_a(\mathbf{k}^T\tau^j\mathbf{k})\left[\frac{G_1^2-G_2^2}{2v_F\Delta}+\left(\frac{iq_0}{v_F\delta q-iq_0}\right)\frac{G_1^2-G_2^2}{2v_F\Delta}\right]\Gamma_{\mu i}\Gamma_{\mu j}$$

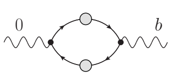
$$-(\tau^i\mathbf{k})_a(\mathbf{q}^T\tau^j\mathbf{k})\left(\frac{iq_0}{v_F\delta q-iq_0}\right)\left(\frac{G_1^2+G_2^2}{2v_F\Delta}\right)i\epsilon_{\mu\nu}\Gamma_{\mu i}\Gamma_{\nu i}\}$$

( $+$ )

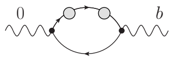
$$=\frac{2e^2}{k_F^4}\int \frac{d^2k}{(2\pi)^2}\frac{1}{\beta}\sum_{k_0}(\mathbf{k}^T\tau^i\mathbf{k})(\mathbf{k}^T\tau^j\mathbf{k})\partial_a\xi_{\mathbf{k}}\partial_b\xi_{\mathbf{k}}\{G_1G_2(G_1^2+G_2^2)+\left(\frac{iq_0}{v_F\delta q-iq_0}\right)\left[G_1G_2(G_1^2+G_2^2)\right.$$

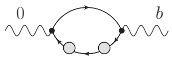
$$\left.-\frac{2G_1G_2}{(2v_F\Delta)^2}\right]\}\Gamma_{\mu i}\Gamma_{\nu j}$$

$$+\frac{4e^2}{k_F^4}\int \frac{d^2k}{(2\pi)^2}\frac{1}{\beta}\sum_{k_0}(\mathbf{q}^T\tau^i\mathbf{k})(\mathbf{k}\tau^j\mathbf{k})\partial_a\xi_{\mathbf{k}}\partial_b\xi_{\mathbf{k}}\left(\frac{iq_0}{(v_F\delta q-iq_0)^2}\right)\left[\frac{G_1^2-G_2^2}{2v_F\Delta}\right]\Gamma_{\mu i}\Gamma_{\nu j}$$

()

$$=\frac{2e^2}{k_F^4}\int \frac{d^2k}{(2\pi)^2}\frac{1}{\beta}\sum_{k_0}(\mathbf{k}^T\tau^i\mathbf{k})(\mathbf{k}^T\tau^j\mathbf{k})\partial_b\xi_{\mathbf{k}}\left[G_1^2G_2^2+\left(\frac{iq_0}{v_F\delta q-iq_0}\right)\frac{G_1^2+G_2^2}{(2v_F\Delta)^2}\right]\Gamma_{\mu i}\Gamma_{\nu j}$$

$\Bigg($ 

$+$ 

$\Bigg)$

$$
\begin{aligned}
=&\frac{2e^2}{k_F^4}\int\frac{d^2k}{(2\pi)^2}\frac{1}{\beta}\sum_{k_0}(\mathbf{k}^T\tau^i\mathbf{k})(\mathbf{k}^T\tau^j\mathbf{k})\partial_b\xi_{\mathbf{k}}\left(\frac{iq_0}{v_F\delta q-iq_0}\right)\left[G_1G_2(G_1^2+G_2^2)-\frac{2G_1G_2}{(2v_F\Delta)^2}\right]\Gamma_{\mu i}\Gamma_{\nu j}\\
&+\frac{4e^2}{k_F^4}\int\frac{d^2k}{(2\pi)^2}\frac{1}{\beta}\sum_{k_0}(\mathbf{q}^T\tau^i\mathbf{k})(\mathbf{k}\tau^j\mathbf{k})\partial_b\xi_{\mathbf{k}}\left(\frac{iq_0}{(v_F\delta q-iq_0)^2}\right)\left[\frac{G_1^2-G_2^2}{2v_F\Delta}\right]\Gamma_{\mu i}\Gamma_{\nu j}
\end{aligned}
$$

From tedius and careful algebra, it can be shown that

$$q_aK_1^{ab}(q)+iq_0K_1^{0b}(q)=0$$

Chapter 5

Conclusions

By using bosonization and diagrammatic methods, some key properties of the system are uncovered. In Chapter Three, the high dimensional bosonization technology was applied to our model. The theory is transformed from the theory of fermionic fluids into a theory of bosons. The interaction becomes a Sine-Gordon type interaction vortex.

Bosonization framework produces similar basic physics to the Hartree-Fock calculations. We showed that from reading off the low-energy effective bosonized Hamiltonian, we can see that it is odd in the order parameters and the **T** symmetry is broken spontaneously. The coefficients of the higher order terms in the dispersion suggest that the correction to the theory arises from them being coupled non-linearly to the curvature of the Fermi surfaces. This suggests that there are two sources of non-linearity in the model. One is from the curvature that governs the Pomeranchuk-type physics, and the other is from the cosine term that governs the physics of the β_1 phase. In fully bosonic interacting theory, the competition between the Sine-Gordon term and the density-density interactions determines whether the system ends up in the β_1 or in the α_1 phase.

The stability condition for the bosonized theory is explored. We found that when $f_l(0)$ is reasonably negative, the system becomes unstable and a curvature effect is needed to stabilize our theory. The stability condition is of the Pomeranchuk type. The correction to the critical point is derived by both methods, bosonization and diagrammatic expansion. Furthermore, for the ground state solutions to be stable, higher order dispersion terms are needed (particularly the cubic order terms). By picking a solution that extremizes the action and by constraining the solution with the intuition from the mean-field results, we can reproduce the behavior of the β_1 phase at the mean-field level. From the solution to the β_1 phase, we classify it by computing the Hall conductance. We learn from it that there is a problem with the bosonization approach. There are two ansatzes for the β_1 phase: a solution that does not capture the density (Eq. (3.110)), and a solution with both phase winding and density. The first gives vanishing Hall conductivity while the latter ansatz (Eq. (3.109)) is not consistent with the mean-field results.

Motivated by the observation that the order parameters for the α_1 phase and the β_1 phase are not compatible, we employ the diagrammatic method to gain more insights. The electromagnetic-coupled fermionic action is revisited. Using the Hubbard-Stratonovich decoupling and then integrating out the fermionic fields, we are left with an effective theory in terms of the nematic fields ($\Gamma_{\mu i}$). Feynman rules for this theory are also established. Operator product expansion (OPE) tools in momentum space are developed. This new technology helps keep the diagrammatic calculations at finite momentum transfer ($\mathbf{q}$) tractible. Full response functional is derived. We obtained polarization tensors with a rich structure. They contained not only the Hall response but also obtained the optical responses that depend on higher harmonics. Furthermore, we find that there is a correction to the Hall response that depends on the mean density of state (DOS) ($\bar{r}$).

5.1 Berry Phase and Hall response

Two terms in the effective action of great interest are: the term that sources the Berry phase, and the Chern-Simons term. They are

$$\begin{aligned}\mathcal{L}_{\text{Berry}} &= -\frac{1}{2}\left(\frac{r}{2v_F\Delta}\right)\epsilon_{\mu\nu}\Gamma_{\mu i}(\partial_t\Gamma_{\nu i}).\\ \mathcal{L}_{\text{CS}} &= \left(\frac{4e^2}{k_F^2}\right)\left(\frac{r}{2v_F\Delta}\right)(\epsilon_{ij}\epsilon_{\mu\nu}\Gamma_{\mu i}\Gamma_{\nu j})\epsilon_{ab}\left[A_a(x)\partial_t A_b(x) + A_0(x)\partial_a A_b(x).\right]\end{aligned} \tag{5.1}$$

The first term is similar to Wess-Zumino term, which gives the dynamic to the $\Gamma_{\mu i}$ fields but in this case the prefactor is unquantized. From the Chern-Simons term, it is clear that the Hall response is unquantized and only non-zero in the β_1 phase when the order parameters are perpendicular to each other (Eq. (4.55)). An interesting fact is that the final form of these terms act like they have a mass gap. This is in some sense not surprising because in integrating out the fermions, one has to pay an energy price. To the best of our knowledge, this is the first time that the full polarization tensor is calculated for a two-band time-reversal broken metallic phase.

5.2 Future Work

Empowered by the full effective action coupled to the electromagnetic field, we look forward to next steps to build off of the theoretical foundation here to study the effective theory in the renormalization group framework.

Furthermore, in Chapter Three it was shown that bosonizing each band independently is not quite correct. The Hamiltonian we obtain does not know about the Berry curvature, which is more of a band effect. Since now we have the form of the polarization tensor, we can reverse engineer to bosonize the correct current that is aware of the mixing of the two bands.

The work here suggests a range of future investigations, both on the theoretical side to refine this approach as well on the applied side in order to enable next-generation materials and quantum devices.

Appendix A

A.1 Bosonization of the interaction Hamiltonian

In this appendix, details of bosonizing H_{int} are presented. The part of the Hamiltonian in question is

$$H_{\text{int}} = \sum_{\mu=x,y} \sum_{\mathbf{q}} \left(\frac{f_l(\mathbf{q})}{2} \right) \Phi_{li\mu}(\mathbf{q}) \Phi_{li\mu}(-\mathbf{q}) \tag{A.1}$$

where

$$\Phi_{l1\mu}(\mathbf{q}) = \sum_{\mathbf{k}} \sum_{m,n} \cos(l\theta_{\mathbf{k}}) : \psi^{\dagger}_{m\mathbf{k}+\mathbf{q}} \sigma^{\mu}_{mn} \psi_{n\mathbf{k}} : \tag{A.2}$$

$$\Phi_{l2\mu}(\mathbf{q}) = \sum_{\mathbf{k}} \sum_{m,n} \sin(l\theta_{\mathbf{k}}) : \psi^{\dagger}_{m\mathbf{k}+\mathbf{q}} \sigma^{\mu}_{mn} \psi_{n\mathbf{k}} : \tag{A.3}$$

and for EM response calculations the following generalized forms are required

$$\Phi_{l1\mu}(\mathbf{q}) = \sum_{\mathbf{k}} \sum_{mn} \left(\frac{\sigma^{\mu}_{mn}}{2k_F^l} \right) \left([k_1 + ik_2]^l + \text{h.c.} \right) : \psi^{\dagger}_{m\mathbf{k}+\mathbf{q}} \psi_{n\mathbf{k}} : \tag{A.4}$$

$$\Phi_{l2\mu}(\mathbf{q}) = \sum_{\mathbf{k}} \sum_{mn} \left(\frac{\sigma^{\mu}_{mn}}{i2k_F^l} \right) \left([k_1 + ik_2]^l - \text{h.c.} \right) : \psi^{\dagger}_{m\mathbf{k}+\mathbf{q}} \psi_{n\mathbf{k}} : \tag{A.5}$$

We will regard that $f_l(\mathbf{q})$ has support only at $\mathbf{q}=0$ and we will mostly specialize to the $l=2$ mode. For reference the analogous real space expressions are

$$\tilde{\Phi}_{li\mu}(\mathbf{x}) = \frac{1}{L^2}\sum_{\mathbf{q}} \mathrm{e}^{i\mathbf{q}\cdot\mathbf{x}}\Phi_{li\mu}(\mathbf{q}) \tag{A.6}$$

$$\tilde{f}_l(\mathbf{r}) = \sum_{\mathbf{q}} f_l(\mathbf{q})\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}} \tag{A.7}$$

$$H_{\text{int}} = \frac{1}{2}\sum_{\mu=x,y}\int d^2x' \int d^2x\, \tilde{f}_l(\mathbf{x}'-\mathbf{x})\tilde{\Phi}_{li\mu}(\mathbf{x}')\tilde{\Phi}_{li\mu}(\mathbf{x}) \tag{A.8}$$

In the case of $l=2$, the nematic fields $\tilde{\Phi}_{li\mu}$ when the $\mathbf{q}=0$ mode dominates has the following form in terms of the patched fermions $\psi_{mS}(\mathbf{x})$

$$\tilde{\Phi}_{2i\mu}(\mathbf{x}) = \sum_S :\psi^\dagger_{mS}(\mathbf{x})\left(\sigma^\mu_{mn}\tau^i_{ab}\frac{(\mathbf{k}_S)_a(\mathbf{k}_S)_b}{k_F^2}\right)\psi_{mS}(\mathbf{x}): \tag{A.9}$$

where the matrices

$$\tau^1 = \sigma^z, \quad \tau^2 = \sigma^x \tag{A.10}$$

are determined by the $l=2$ partial wave. The form above is needed for minimal coupling to external EM fields. Otherwise for the situations without EM fields, we have the following real-space forms of the $\tilde{\Phi}_{li\mu}$ fields

$$\tilde{\Phi}_{l1\mu}(\mathbf{x}) = \sum_S \cos(l\theta_S) :\psi^\dagger_{mS}(\mathbf{x})\sigma^\mu_{mn}\psi_{nS}(\mathbf{x}): \tag{A.11}$$

$$\tilde{\Phi}_{l2\mu}(\mathbf{x}) = \sum_S \sin(l\theta_S) :\psi^\dagger_{mS}(\mathbf{x})\sigma^\mu_{mn}\psi_{nS}(\mathbf{x}): \tag{A.12}$$

This then yields

$$
\begin{aligned}
H_{\text{int}} &= \sum_{\mu=x,y} \int d^2x \int d^2x' \ \left[\tfrac{\tilde{f}_l(\mathbf{x}'-\mathbf{x})}{2}\right] \ \tilde{\Phi}_{li\mu}(\mathbf{x}')\tilde{\Phi}_{li\mu}(\mathbf{x}) \\
&= \sum_{\mu=x,y} \int d^2x' \int d^2x \left[\tfrac{\tilde{f}_l(\mathbf{x}'-\mathbf{x})}{2}\right] \sum_{S,T} \Big\{ \cos(l\theta_S)\cos(l\theta_T) : \psi^\dagger_S(\mathbf{x}')\sigma^\mu\psi_S(\mathbf{x}') :: \psi^\dagger_T(\mathbf{x})\sigma^\mu\psi_T(\mathbf{x}) : \\
&\qquad \sin(l\theta_S)\sin(l\theta_T) : \psi^\dagger_S(\mathbf{x}')\sigma^\mu\psi_S(\mathbf{x}') :: \psi^\dagger_T(\mathbf{x})\sigma^\mu\psi_T(\mathbf{x}) : \Big\} \\
&= \sum_{\mu=x,y} \int d^2x' \int d^2x \left[\tfrac{\tilde{f}_l(\mathbf{x}'-\mathbf{x})}{2}\right] \sum_{S,T} \cos[l(\theta_S-\theta_T)] : \psi^\dagger_S(\mathbf{x}')\sigma^\mu\psi_S(\mathbf{x}') :: \psi^\dagger_T(\mathbf{x})\sigma^\mu\psi_T(\mathbf{x}) :
\end{aligned}
\tag{A.13}
$$

Next the fermion bilinears are explicitly

$$
\psi^\dagger_{mS}(\mathbf{x})\sigma^x_{mn}\psi_{nS}(\mathbf{x}) = \psi^\dagger_{1S}(\mathbf{x})\psi_{2S}(\mathbf{x}) + \psi^\dagger_{2S}(\mathbf{x})\psi_{1S}(\mathbf{x}) \tag{A.14}
$$

$$
\psi^\dagger_{mS}(\mathbf{x})\sigma^y_{mn}\psi_{nS}(\mathbf{x}) = -i\psi^\dagger_{1S}(\mathbf{x})\psi_{2S}(\mathbf{x}) + i\psi^\dagger_{2S}(\mathbf{x})\psi_{1S}(\mathbf{x}). \tag{A.15}
$$

Due to the normal ordering, we may consider the case when $S \neq T$ as independent fermion bilinears since the two pairs of fermion bilinears commute. The intra-patch case when $S = T$, has to be considered separately.

First when $S \neq T$, we may apply the bosonization formulae independently to each bilinear and using

$$
\psi^\dagger_{mS}(\mathbf{x})\psi_{nS}(\mathbf{x}) \rightsquigarrow \eta_{mS}(x_t)\eta_{nS}(x_t)(\hbar v_F N(0)\lambda) : \mathrm{e}^{i[\phi_{mS}(\mathbf{x})-\phi_{nS}(\mathbf{x})]} : \quad \text{where} \quad x_t \equiv \vec{n}_S \cdot \mathbf{x} \tag{A.16}
$$

yields

$$
\begin{aligned}
\psi^\dagger_{mS}(\mathbf{x})\sigma^x_{mn}\psi_{nS}(\mathbf{x}) &\rightsquigarrow (\hbar v_F N(0)\lambda)\eta_{1S}(x_t)\eta_{2S}(x_t) : \left(\mathrm{e}^{i[\phi_{1S}(\mathbf{x})-\phi_{2S}(\mathbf{x})]} - \mathrm{e}^{i[\phi_{2S}(\mathbf{x})-\phi_{1S}(\mathbf{x})]}\right) : \\
&= +2i[\hbar v_F N(0)\lambda]\eta_{1S}(x_t)\eta_{2S}(x_t) : \sin[\phi_{1S}(\mathbf{x}) - \phi_{2S}(\mathbf{x})] :
\end{aligned}
\tag{A.17}
$$

where a negative sign comes from anti-commuting the Klein factors. Similarly we have also

$$\psi^\dagger_{mS}(\mathbf{x})\sigma^y_{mn}\psi_{nS}(\mathbf{x}) \rightsquigarrow (\hbar v_F N(0)\lambda)\eta_{1S}(x_t)\eta_{2S}(x_t) : \left(-i\mathrm{e}^{i[\phi_{1S}(\mathbf{x})-i\phi_{2S}(\mathbf{x})]} - i\mathrm{e}^{i[\phi_{2S}(\mathbf{x})-\phi_{1S}(\mathbf{x})]}\right) :$$
$$= -2i[\hbar v_F N(0)\lambda]\eta_{1S}(x_t)\eta_{2S}(x_t) : \cos[\phi_{1S}(\mathbf{x}) - \phi_{2S}(\mathbf{x})] : \qquad \text{(A.18)}$$

Hence for the terms where $S \neq T$, we arrive at a contribution of

$$\int d^2x' \int d^2x \left[\tfrac{\tilde{f}_l(\mathbf{x}'-\mathbf{x})}{2}\right] \sum_{S\neq T} \cos[l(\theta_S - \theta_T)][2\hbar v_F N(0)\lambda]^2[i\eta_{1S}(x'_t)\eta_{2S}(x'_t)][i\eta_{1T}(x_t)\eta_{2T}(x_t)]$$
$$\times \{: \sin[\Delta\phi_S(\mathbf{x}')]\sin[\Delta\phi_T(\mathbf{x})] : + : \cos[\Delta\phi_S(\mathbf{x}')]\cos[\Delta\phi_T(\mathbf{x})] :\}$$

$$= 2[\hbar v_F N(0)\lambda]^2 \int d^2x' \int d^2x \tilde{f}_l(\mathbf{x}' - \mathbf{x}) \sum_{S\neq T} \cos[l(\theta_S - \theta_T)]\, \{: \sin[\Delta\phi_S(\mathbf{x}')]\sin[\Delta\phi_T(\mathbf{x})] : \qquad \text{(A.19)}$$

$$+ : \cos[\Delta\phi_S(\mathbf{x}')]\cos[\Delta\phi_T(\mathbf{x})] :\} \qquad \text{(A.20)}$$

where $\Delta\phi_S(\mathbf{x}) := \phi_{1S}(\mathbf{x}) - \phi_{2S}(\mathbf{x})$ and we have set the **unitary operators** $(i\eta_{1S,T}\eta_{2S,T})$ **to one** in the final line since they commute with the Hamiltonian[1].

In the case where $S = T$ the manipulations above should apply up until the point when $\mathbf{x}' = \mathbf{x}$ where an additional operator may be generated as a result of an operator product expansion as $\mathbf{x}' \to \mathbf{x}$. First we use the δ-function approximation to $\tilde{f}(\mathbf{r}) \approx f_l(0)L^2\delta(\mathbf{r})$ in the limit of small κ_l

$$\left(\frac{f_l(0)L^2}{2}\right) \sum_{\mu=x,y} \int d^2x \sum_S : \psi^\dagger_S(\mathbf{x})\sigma^\mu\psi_S(\mathbf{x}) :: \psi^\dagger_S(\mathbf{x})\sigma^\mu\psi_S(\mathbf{x}) : . \qquad \text{(A.21)}$$

[1]In the interpretation(approximation) that the Klein factors are Majorana operators.

Then applying the same bosonization formulas yields

$$2f_l(0)[\hbar v_F N(0)\lambda L]^2 \int d^2x \sum_S \{: \sin[\Delta\phi_S(\mathbf{x})] :: \sin[\Delta\phi_S(\mathbf{x})] : + : \cos[\Delta\phi_S(\mathbf{x})] :: \cos[\Delta\phi_S(\mathbf{x})] :\} \tag{A.22}$$

where the products at coincident points now need an OPE. Now using the standard BCH identity, Eq. (3.37)

$$\begin{aligned}
: \sin[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})] :: \sin[\Delta\phi(\mathbf{x})] : \quad &= -\left(\frac{1}{4}\right) : \left(\mathrm{e}^{i\Delta\phi_S(\mathbf{x}+\vec{\epsilon})} - \mathrm{e}^{-i\Delta\phi_S(\mathbf{x}+\vec{\epsilon})}\right) :: \left(\mathrm{e}^{i\Delta\phi_S(\mathbf{x})} - \mathrm{e}^{-i\Delta\phi_S(\mathbf{x})}\right) : \\
&= -\left(\frac{1}{4}\right)\big\{: \mathrm{e}^{i\Delta\phi_S(\mathbf{x}+\vec{\epsilon})} :: \mathrm{e}^{i\Delta\phi_S(\mathbf{x})} : - : \mathrm{e}^{i\Delta\phi_S(\mathbf{x}+\vec{\epsilon})} :: \mathrm{e}^{-i\Delta\phi_S(\mathbf{x})} : \\
&\quad - : \mathrm{e}^{-i\Delta\phi_S(\mathbf{x}+\vec{\epsilon})} :: \mathrm{e}^{i\Delta\phi_S(\mathbf{x})} : + : \mathrm{e}^{-i\Delta\phi_S(\mathbf{x}+\vec{\epsilon})} :: \mathrm{e}^{-i\Delta\phi_S(\mathbf{x})} :\big\} \\
&= \left(\frac{1}{4}\right)\big\{- : \mathrm{e}^{+i[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})+\Delta\phi_S(\mathbf{x})]} : \mathrm{e}^{-\langle\Delta\phi_S(\mathbf{x}+\vec{\epsilon})\Delta\phi_S(\mathbf{x})\rangle} \\
&\quad + : \mathrm{e}^{+i[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})-\Delta\phi_S(\mathbf{x})]} : \mathrm{e}^{+\langle\Delta\phi_S(\mathbf{x}+\vec{\epsilon})\Delta\phi_S(\mathbf{x})\rangle} \\
&\quad + : \mathrm{e}^{-i[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})-\Delta\phi_S(\mathbf{x})]} : \mathrm{e}^{+\langle\Delta\phi_S(\mathbf{x}+\vec{\epsilon})\Delta\phi_S(\mathbf{x})\rangle} \\
&\quad - : \mathrm{e}^{-i[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})+\Delta\phi_S(\mathbf{x})]} : \mathrm{e}^{-\langle\Delta\phi_S(\mathbf{x}+\vec{\epsilon})\Delta\phi_S(\mathbf{x})\rangle}\big\}
\end{aligned} \tag{A.23}$$

And by exchanging some signs

$$\begin{aligned}
: \cos[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})] :: \cos[\Delta\phi(\mathbf{x})] : \quad &= \left(\frac{1}{4}\right)\big\{+ : \mathrm{e}^{+i[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})+\Delta\phi_S(\mathbf{x})]} : \mathrm{e}^{-\langle\Delta\phi_S(\mathbf{x}+\vec{\epsilon})\Delta\phi_S(\mathbf{x})\rangle} \\
&\quad + : \mathrm{e}^{+i[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})-\Delta\phi_S(\mathbf{x})]} : \mathrm{e}^{+\langle\Delta\phi_S(\mathbf{x}+\vec{\epsilon})\Delta\phi_S(\mathbf{x})\rangle} \\
&\quad + : \mathrm{e}^{-i[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})-\Delta\phi_S(\mathbf{x})]} : \mathrm{e}^{+\langle\Delta\phi_S(\mathbf{x}+\vec{\epsilon})\Delta\phi_S(\mathbf{x})\rangle} \\
&\quad + : \mathrm{e}^{-i[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})+\Delta\phi_S(\mathbf{x})]} : \mathrm{e}^{-\langle\Delta\phi_S(\mathbf{x}+\vec{\epsilon})\Delta\phi_S(\mathbf{x})\rangle}\big\}
\end{aligned} \tag{A.24}$$

Hence from some cancellations

$$
\begin{aligned}
&: \sin[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})] :: \sin[\Delta\phi_S(\mathbf{x})] : + : \cos[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})] :: \cos[\Delta\phi_S(\mathbf{x})] : \\
&= \left(\frac{1}{2}\right)\left\{: \mathrm{e}^{i[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})-\Delta\phi_S(\mathbf{x})]} : + : \mathrm{e}^{-i[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})-\Delta\phi_S(\mathbf{x})]} \!:\! \mathrm{e}^{\langle\Delta\phi_S(\mathbf{x}+\vec{\epsilon})\Delta\phi_S(\mathbf{x})\rangle}\right\} \\
&= : \cos\underbrace{[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})-\Delta\phi_S(\mathbf{x})] :}_{=\epsilon\nabla_S(\Delta\phi)(\mathbf{x})+O(\epsilon^2)} \qquad \times \qquad \underbrace{\mathrm{e}^{\langle\Delta\phi_S(\mathbf{x}+\vec{\epsilon})\Delta\phi_S(\mathbf{x})\rangle}}_{=\mathrm{e}^{\langle\phi_{1S}(\mathbf{x}+\vec{\epsilon})\phi_{1S}(\mathbf{x})\rangle}\mathrm{e}^{\langle\phi_{2S}(\mathbf{x}+\vec{\epsilon})\phi_{2S(\mathbf{x})}\rangle}} \\
&=: \left(1-\tfrac{\epsilon^2}{2}[\Delta\phi'(\mathbf{x})]^2+O(\epsilon^4)\right) : \left(\frac{i}{\lambda\epsilon}\right)^2
\end{aligned}
\tag{A.25}
$$

Finally taking only the first non-divergent terms as dictated by the normal ordering leads to

$$
\begin{aligned}
&: \sin[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})] :: \sin[\Delta\phi_S(\mathbf{x})] : + : \cos[\Delta\phi_S(\mathbf{x}+\vec{\epsilon})] :: \cos[\Delta\phi_S(\mathbf{x})] : \\
&= \left(\frac{1}{2\lambda^2}\right)[\nabla_S\phi_{1S}(\mathbf{x})-\nabla_S\phi_{2S}(\mathbf{x})]^2
\end{aligned}
\tag{A.26}
$$

which yields the final term

$$
[\hbar v_F N(0)]^2 f_l(0) L^2 \int d^2x \sum_S [\nabla_S\phi_{1S}(\mathbf{x})-\nabla_S\phi_{2S}(\mathbf{x})]^2 \tag{A.27}
$$

Thus the full bosonized interaction Hamiltonian H_{int} is

$$
\boxed{
\begin{aligned}
H_{\text{int}} &= 2[\hbar v_F N(0)\lambda]^2 \int d^2x' \int d^2x\ \tilde{f}_l(\mathbf{x}'-\mathbf{x}) \sum_{S,T} \cos[l(\theta_S-\theta_T)] : \cos[\Delta\phi_S(\mathbf{x}')-\Delta\phi_T(\mathbf{x})] : \\
&\quad + [\hbar v_F N(0)]^2 f_l(0) L^2 \int d^2x \sum_S [\nabla_S\phi_{1S}(\mathbf{x})-\nabla_S\phi_{2S}(\mathbf{x})]^2
\end{aligned}
}
\tag{A.28}
$$

where the first term has been consolidated using a standard trigonometric identity.

A.1.1 Hubbard Stratonovich Fields

For the purposes of comparison with previous mean-field results [Sun and Fradkin, 2008, Wu et al., 2007] and as a theoretical device, we will employ the use of Hubbard-Stratonovich (HS)mean-fields that decouple – in the sense of functional integrals – the quartic terms in H_{int}. Initially this was used as a convenience to bosonize H_{int} but becomes a necessary tool in studying mean-field phases where $\phi_{li\mu} := \langle\tilde{\Phi}_{li\mu}\rangle$ becomes a quantity of interest.

We write

$$\begin{aligned} H_{\text{int}} = &\sum_{\mu=x,y}\int d^2x' \int d^2x \, \left[\tfrac{\tilde{f}_l(\mathbf{x}'-\mathbf{x})}{2}\right]\left(\phi_{li\mu}(\mathbf{x}')\tilde{\Phi}_{li\mu}(\mathbf{x}) + \phi_{li\mu}(\mathbf{x})\tilde{\Phi}_{li\mu}(\mathbf{x}')\right) \\ &- \sum_{\mu=x,y}\int d^2x' \int d^2x \, \left[\tfrac{\tilde{f}_l(\mathbf{x}'-\mathbf{x})}{2}\right]\phi_{li\mu}(\mathbf{x}')\phi_{li\mu}(\mathbf{x}) \end{aligned} \tag{A.29}$$

where $\phi_{li\mu}$ is the HS field which when functionally integrated produces the original H_{int}. At a saddle point solution (least action) it has to satisfy

$$\langle\tilde{\Phi}_{li\mu}(\mathbf{x})\rangle = \phi_{li\mu}(\mathbf{x}) \tag{A.30}$$

such that $\phi_{li\mu}$ is a mean field for a quantum effective potential(action). The nematic field $\tilde{\Phi}_{li\mu}(\mathbf{x})$ has previously been bosonized, but can be neatly arranged in column vectors as

$$\vec{\tilde{\Phi}}_{l1}(\mathbf{x}) = [2\hbar v_F N(0)\lambda]\sum_S [i\eta_{1S}(x_t)\eta_{2S}(x_t)]\cos(l\theta_S)\begin{pmatrix} :\sin[\Delta\phi_S(\mathbf{x})]: \\ -:\cos[\Delta\phi_S(\mathbf{x})]: \end{pmatrix} \tag{A.31}$$

$$\vec{\tilde{\Phi}}_{l2}(\mathbf{x}) = [2\hbar v_F N(0)\lambda]\sum_S [i\eta_{1S}(x_t)\eta_{2S}(x_t)]\cos(l\theta_S)\begin{pmatrix} :\cos[\Delta\phi_S(\mathbf{x})]: \\ -:\cos[\Delta\phi_S(\mathbf{x})]: \end{pmatrix} \tag{A.32}$$

A.2 Angular Expansions

Here we list some more complicated angular expansions needed for the computation of Feynman diagrams. They take the form as expansions in terms of the tensors δ_{ab}, ϵ_{ab}, $\tau^1_{ab} \equiv \sigma^z_{ab}$, $\tau^2_{ab} \equiv \sigma^x_{ab}$, and the unit vectors $\hat{\mathbf{k}} = (\cos\theta_k, \sin\theta_k)$, $\hat{\mathbf{q}} = (\cos\theta_q, \sin\theta_q)$.

$$
\begin{aligned}
[A] \quad (\tau^i\hat{\mathbf{k}})_a(\tau^j\hat{\mathbf{k}})_b = \frac{1}{2}\big(&\delta_{ij}\delta_{ab} + \epsilon_{ij}\epsilon_{ab} + \sigma^z_{ij}\left[\tau^1_{ab}\cos 2\theta_k - \tau^2_{ab}\sin 2\theta_k\right] \\
&+\sigma^x_{ij}\left[\tau^1_{ab}\sin 2\theta_k + \tau^2_{ab}\cos 2\theta_k\right]\big)
\end{aligned}
$$

$$
\begin{aligned}
[B] \quad (\hat{\mathbf{k}}^T\tau^i\hat{\mathbf{k}})(\tau^j\hat{\mathbf{k}})_a\hat{k}_b = &\frac{1}{4}\left(\delta_{ij}\delta_{ab} - \epsilon_{ab}\epsilon_{ij}\right) \\
&+ \frac{1}{4}\big(\delta_{ij}\left[\tau^1_{ab}\cos 2\theta_k + \tau^2_{ab}\sin 2\theta_k\right] - \epsilon_{ij}\left[\tau^1_{ab}\sin 2\theta_k - \tau^2_{ab}\cos 2\theta_k\right]\big) \\
&+ \frac{1}{4}\sigma^z_{ij}\left[\delta_{ab}\cos 4\theta_k + \epsilon_{ab}\sin 4\theta_k + \tau^1_{ab}\cos 2\theta_k - \tau^2_{ab}\sin 2\theta_k\right] \\
&+ \frac{1}{4}\sigma^x_{ij}\left[\delta_{ab}\sin 4\theta_k - \epsilon_{ab}\cos 4\theta_k + \tau^1_{ab}\sin 2\theta_k + \tau^2_{ab}\cos 2\theta_k\right]
\end{aligned}
$$

$$
[C] \quad \hat{\mathbf{q}}\cdot\hat{\mathbf{k}} = \cos(\theta_k - \theta_q)
$$

$$
[D] \quad (\hat{\mathbf{q}}^T\tau^1\hat{\mathbf{k}}) = \cos(\theta_k + \theta_q), \qquad (\hat{\mathbf{q}}^T\tau^2\hat{\mathbf{k}}) = \sin(\theta_k + \theta_q)
$$

$$[E] \quad \hat{\mathbf{q}}_a \hat{\mathbf{k}}_b = \frac{1}{2}\left[\delta_{ab}\cos(\theta_k - \theta_q) + \epsilon_{ab}\sin(\theta_k - \theta_q) + \tau^1_{ab}\cos(\theta_k + \theta_q) + \tau^2_{ab}\sin(\theta_k + \theta_q)\right]$$

$$\begin{aligned}[F] \quad (\tau^1\hat{\mathbf{q}})_a \hat{k}_b = \frac{1}{2}\Big[&\delta_{ab}\cos(\theta_k + \theta_q) + \epsilon_{ab}\sin(\theta_k + \theta_q) + \tau^1_{ab}\cos(\theta_k - \theta_q) \\ &+\tau^2_{ab}\sin(\theta_k - \theta_q)\Big]\end{aligned}$$

$$\begin{aligned}[G] \quad (\tau^2\hat{\mathbf{q}})_a \hat{k}_b = \frac{1}{2}\Big[&\delta_{ab}\sin(\theta_k + \theta_q) - \epsilon_{ab}\cos(\theta_k + \theta_q) - \tau^1_{ab}\sin(\theta_k - \theta_q) \\ &+\tau^2_{ab}\cos(\theta_k - \theta_q)\Big]\end{aligned}$$

$$[H] \quad (\hat{\mathbf{q}}^T\tau^i\hat{\mathbf{k}})(\tau^j\hat{\mathbf{k}})_1 = \frac{1}{2}\left[\delta_{ij}\cos\theta_q - \epsilon_{ij}\sin\theta_q + \sigma^z_{ij}\cos(2\theta_k + \theta_q) + \sigma^x_{ij}\sin(2\theta_k + \theta_q)\right]$$

$$[I] \quad (\hat{\mathbf{q}}^T\tau^i\hat{\mathbf{k}})(\tau^j\hat{\mathbf{k}})_2 = \frac{1}{2}\left[\delta_{ij}\sin\theta_q + \epsilon_{ij}\cos\theta_q - \sigma^z_{ij}\sin(2\theta_k + \theta_q) + \sigma^x_{ij}\cos(2\theta_k + \theta_q)\right]$$

$$
\begin{aligned}
[J] \quad (\hat{\mathbf{q}}^T \tau^i \hat{\mathbf{k}})(\tau^j \hat{\mathbf{k}})_a \hat{k}_b = \frac{1}{4} \{ & [\delta_{ij}\delta_{ab} - \epsilon_{ij}\epsilon_{ab}] \cos(\theta_k - \theta_q) + [\delta_{ij}\epsilon_{ab} + \epsilon_{ij}\delta_{ab}] \sin(\theta_k - \theta_q) \\
& + [\delta_{ij}\tau^1_{ab} + \epsilon_{ij}\tau^2_{ab} + \sigma^z_{ij}\tau^1_{ab} + \sigma^x_{ij}\tau^2_{ab}] \cos(\theta_k + \theta_q) \\
& + [-\epsilon_{ij}\tau^1_{ab} + \delta_{ij}\tau^2_{ab} - \sigma^z_{ij}\tau^2_{ab} + \sigma^x_{ij}\tau^1_{ab}] \sin(\theta_k + \theta_q) \\
& + [\sigma^z_{ij}\delta_{ab} - \sigma^x_{ij}\epsilon_{ab}] \cos(3\theta_k + \theta_q) \\
& + [\sigma^z_{ij}\epsilon_{ab} + \sigma^z_{ij}\delta_{ab}] \sin(3\theta_k + \theta_q) \}
\end{aligned}
$$

$$
[K] \quad (\hat{\mathbf{k}}^T \tau^i \hat{\mathbf{k}})(\hat{\mathbf{k}}^T \tau^j \hat{\mathbf{k}}) = \frac{1}{2}\delta_{ij} + \frac{\sigma^z_{ij}}{2} \cos 4\theta_k + \frac{\sigma^x_{ij}}{2} \sin 4\theta_k
$$

A.3 Harmonic Expansion of the FL density fluctuation propagator

Calculations will often involve integrals with the function $(v_F\delta q - iq_0)^{-1}$, with $\delta q = |\mathbf{q}|\cos\theta_{kq}$. We present here the (angular) harmonic expansion in 2D of this function and a related one which is more commonly associated with a Green's function of density fluctuations near the Fermi-surface. Defining the following geometrical quantities

$$\|q\| := \sqrt{q_0^2 + v_F^2|\mathbf{q}|}, \quad x := \frac{q_0}{\|q\|}, \quad y := \frac{v_F|\mathbf{q}|}{\|q\|}, \quad \tan\alpha := \frac{y}{|x|} \Rightarrow \tan(\tfrac{\alpha}{2}) = \frac{y}{1+|x|},$$
$$\theta_{kq} := \theta_k - \theta_q$$

we have

$$[A] \quad \frac{1}{v_F\delta q - iq_0} = \frac{i\,\mathrm{sign}(q_0)}{\|q\|}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n \mathrm{e}^{i\frac{n\pi}{2}\mathrm{sign}(q_0)}\tan^n\left(\frac{\alpha}{2}\right)\cos(n\theta_{kq})\right)$$

$$[B] \quad \frac{v_F\delta q}{v_F\delta q - iq_0} = 1 - \frac{|q_0|}{\|q\|}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n \mathrm{e}^{i\frac{n\pi}{2}\mathrm{sign}(q_0)}\tan^n\left(\frac{\alpha}{2}\right)\cos(n\theta_{kq})\right)$$

where $|\tan(\frac{\alpha}{2})| < 1$ such that the series is absolutely convergent. The second function is the one more often encountered when computing the Lindhard function associated to the density-density correlator. Expanding in $q_0/(v_F|\mathbf{q}|)$ gives the usual Landau damping form.

This useful expansion is derived in the following. The basic idea is to use the Schwinger "proper time" trick to turn the fraction to an integral over an exponential, which we then expand using the Jacobi-Anger identities. First write

$$
\begin{aligned}
\frac{1}{v_F \delta q - i q_0} &= \left(\frac{1}{\|q\|}\right) \frac{1}{y \cos\theta_{kq} - ix} \\
&= \frac{i\mathrm{sign}(q_0)}{\|q\|} \int_0^\infty \mathrm{d}s \; \mathrm{e}^{-|x|s} \mathrm{e}^{-i\mathrm{sign}(x) ys \cos\theta_{kq}} \\
&= \frac{i\mathrm{sign}(q_0)}{\|q\|} \int_0^\infty \mathrm{d}s \; \mathrm{e}^{-|x|s} \sum_{n=0}^{\infty} i^n J_n(-\mathrm{sign}(x) ys) \, \mathrm{e}^{in\theta_{kq}} \\
&= \frac{i\mathrm{sign}(q_0)}{\|q\|} \int_0^\infty \mathrm{d}s \; \mathrm{e}^{-|x|s} \left[J_0(ys) + 2 \sum_{n=1}^{\infty} i^n J_n \left(-\mathrm{sign}(x) ys\right) \cos(n\theta_{kq}) \right] \\
&= \frac{i\mathrm{sign}(q_0)}{\|q\|} \left[\int_0^\infty \mathrm{e}^{-|x|s} J_0(ys) \mathrm{d}s + 2 \sum_{n=1}^{\infty} i^n \cos(n\theta_{kq}) \int_0^\infty \mathrm{e}^{-|x|s} J_n \left(-\mathrm{sign}(x) ys\right) \mathrm{d}s \right]
\end{aligned}
$$

where the integrals $\int \mathrm{d}s$ are recognized to be Laplace transforms of the Bessel J_n functions. These transforms have known closed form solutions and using the fact $x^2 + y^2 = 1$, we have the simplifications

$$
\int_0^\infty \mathrm{d}s \; \mathrm{e}^{-|x|s} J_0(ys) = 1
$$

$$
\int_0^\infty \mathrm{d}s \; \mathrm{e}^{-|x|s} J_n \left(-\mathrm{sign}(x) ys\right) = \mathrm{e}^{\frac{in\pi}{2}(1+\mathrm{sign}(x))} \left(\frac{y}{1+|x|}\right)^n
$$

where

$$
\frac{y}{1+|x|} = \frac{v_F \delta q}{\|q\| + |q_0|} < 1
$$

is an $O(1)$ geometrical factor. By kindergarten trigonometry, it is related to $\tan\alpha = y/|x|$ by drawing the unit circle and appropriate chords

$$
\frac{y}{1+|x|} = \tan\left(\frac{\alpha}{2}\right). \tag{A.33}
$$

Combining these expressions yields the expansion [A] and hence [B] as claimed.

A.4 Gradient expansions

In this appendix we describe some simplifications that are encountered in the course of deriving the small q or gradient expansion of the perturbative response kernels. In some instances the expansion resembles that of an Operator Product Expansion of coincident poles in complex frequency space as $q_0 \to 0$. We describe first a simple example of the response bubble encountered in the density-density response.

A.4.1 Density-density example

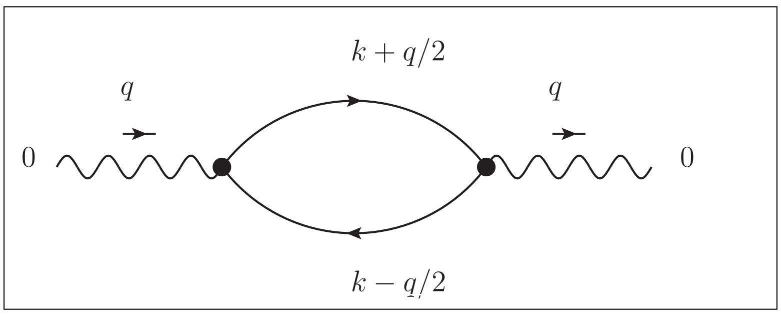


Figure A.1: The simplest density-density response bubble

In calculating the bare or conventional (gapless) density-density response (shown in Figure.A.1) we encounter the following expression obtained from linearizing $\xi_{\mathbf{k}\pm\mathbf{q}/2} = \xi_{\mathbf{k}} \pm v_F \delta q/2$ and Taylor expanding n_F to linear order,

$$\begin{aligned}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\, \frac{1}{\beta} \sum_{k_0} G(ik_0^+, \mathbf{k}^+) G(ik_0^-, \mathbf{k}^-) &= \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \left(\frac{v_F \delta q}{v_F \delta q - i q_0} \right) n_F'(\xi_{\mathbf{k}}) \\ &= \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \left(\frac{v_F \delta q}{v_F \delta q - i q_0} \right) \frac{1}{\beta} \sum_{k_0} G(ik_0, \mathbf{k})^2\end{aligned}$$

where in the last line we have used the relation between Matsubara summed powers of the (bare) Green's function and derivatives of n_F

$$\frac{1}{\beta} \sum_{k_0} G(ik_0, \mathbf{k})^m = \frac{1}{(m-1)!}\, n_F^{(m-1)}(\xi_{\mathbf{k}}), \quad m > 0.$$

Also $\delta q := \mathbf{q}\cdot\hat{\mathbf{k}}$ and $G(z,\mathbf{k}) = (z-\xi_{\mathbf{k}})^{-1}$ is the single particle Green's function. This suggest an identity that is valid only under the Matsubara frequency sum

$$G(ik_0^+,\mathbf{k}^+)G(ik_0^-,\mathbf{k}^-) \equiv \left(\frac{v_F\delta q}{v_F\delta q - iq_0}\right) G(ik_0,\mathbf{k})^2$$

and in the limit of $q \to 0$. Thus in this sense it resembles an OPE in complex frequency space, where in the limit $q \to 0$, the complex energy poles of the G's coincide to form a double pole. Recall 1 also that strictly speaking because the GG integrand represents gapless excitations, a conventional Taylor expansion in q is meaningless, and so is a Laurent expansion in $1/q_0$ or $1/|\mathbf{q}|$. Rather the q-expansion should really be thought of as an OPE like expansion between Green's functions as the operators and with the leading order singularities reflecting the gapless excitations which in this instance are the density modes. Moreover in performing the Matsubara sum, a contour integral is invoked which furthers the analogy with OPE's encountered in CFTs. However opposite to short distance expansions, a small q expansion is a long wavelength, low frequency expansion.

In deriving gradient expansion terms of the response kernel, we will need more replacement identity between Green's functions under the Matsubara sum. This we will derive next and present just the outline of basic steps. However the method generalizes to more complicated products of Green's functions or "OPE-relations". Consider a meromorphic function $F(z)$ which is never singular at $z = k_0$ and the following summation

$$\begin{aligned}
&\frac{1}{\beta}\sum_{k_0} G(ik_0^+,\mathbf{k}^+)G(ik_0^-,\mathbf{k}^-)F(ik_0) \\
&= \oint_{-\mathcal{C}} \frac{\mathrm{d}z}{i2\pi} G(z+iq_0 2,\mathbf{k}^+)G(z-\frac{iq_0}{2},\mathbf{k}^-)F(z)n_F(z) \\
&= \oint_{-\mathcal{C}} \frac{\mathrm{d}z}{i2\pi}\left(\frac{v_F\delta q}{v_F\delta q - iq_0}\right)\left[G(z+\frac{iq_0}{2},\mathbf{k}^+) - G(z-\frac{iq_0}{2},\mathbf{k}^-)\right]F(z)n_F(z) \\
&= \frac{1}{v_F\delta q - iq_0}\oint_{-\mathcal{C}} \frac{\mathrm{d}z}{i2\pi}\left(\frac{1}{z+i\frac{q_0}{2}-\xi_{\mathbf{k}^+}} - \frac{1}{z-i\frac{q_0}{2}-\xi_{\mathbf{k}^-}}\right)F(z)n_F(z)
\end{aligned}$$

where $\mathcal{C}$ is the contour which surrounds (counterclockwise) the poles of $n_F(z)$ located at the odd Matsubara frequencies $z = ik_0 = (2n+1)\pi/k_BT$. Let $(-\mathcal{C})'$ be the contour $(-\mathcal{C})$ but which excludes a neighbourhood which contains $\xi_{\mathbf{k}^\pm} \mp i\frac{q_0}{2}$. Then from linearizing by $(q_0, \mathbf{q})$

$$
\begin{aligned}
&\frac{1}{\beta}\sum_{k_0} G(ik_0^+, \mathbf{k}^+)G(ik_0^-, \mathbf{k}^-)F(ik_0) \\
&= \frac{1}{v_F\delta q - iq_0}\left\{F(\xi_{\mathbf{k}^+} - i\tfrac{q_0}{2})n_F(\xi_{\mathbf{k}^+}) - F(\xi_{\mathbf{k}^-} + i\tfrac{q_0}{2})n_F(\xi_{\mathbf{k}^-})\right\} \\
&+ \frac{1}{v_F\delta q - iq_0}\oint_{-\mathcal{C}'}\frac{\mathrm{d}z}{i2\pi}\left[G(z + i\tfrac{q_0}{2}, \mathbf{k}^+) - G(z - i\tfrac{q_0}{2}, \mathbf{k}^-)\right]F(z)n_F(z) \\
&= F'(\xi_{\mathbf{k}})n_F(\xi_{\mathbf{k}}) + \left(\frac{v_F\delta q}{v_F\delta q - iq_0}\right)F(\xi_{\mathbf{k}})n_F'(\xi_{\mathbf{k}}) \\
&+ \left(\frac{1}{v_F\delta q - iq_0}\right)\oint_{-\mathcal{C}'}\frac{\mathrm{d}z}{i2\pi}\left[G(z + i\tfrac{q_0}{2}, \mathbf{k}^+) - G(z - i\tfrac{q_0}{2}, \mathbf{k}^-)\right]F(z)n_F(z) \\
&= F'(\xi_{\mathbf{k}})n_F(\xi_{\mathbf{k}}) + \left(\frac{v_F\delta q}{v_F\delta q - iq_0}\right)F(\xi_{\mathbf{k}})n_F'(\xi_{\mathbf{k}}) \\
&+ \left(\frac{1}{v_F\delta q - iq_0}\right)\oint_{-\mathcal{C}'}\frac{\mathrm{d}z}{i2\pi}(v_F\delta q - iq_0)G(z, \mathbf{k})^2F(z)n_F(z) \\
&= F'(\xi_{\mathbf{k}})n_F(\xi_{\mathbf{k}}) + F(\xi_{\mathbf{k}})n_F'(\xi_{\mathbf{k}}) + \left(\frac{v_F\delta q}{v_F\delta q - iq_0} - 1\right)F(\xi_{\mathbf{k}})n_F'(\xi_{\mathbf{k}}) \\
&+ \oint_{-\mathcal{C}'}\frac{\mathrm{d}z}{i2\pi}G(z, \mathbf{k})^2F(z)n_F(z) \\
&= \left(\frac{iq_0}{v_F\delta q - iq_0}\right)F(\xi_{\mathbf{k}})n_F'(\xi_{\mathbf{k}}) + \oint_{-\mathcal{C}}\frac{\mathrm{d}z}{i2\pi}G(z, \mathbf{k})^2F(z)n_F(z) \\
&= \oint_{-\mathcal{C}}\frac{\mathrm{d}z}{i2\pi}\left[\frac{iq_0F(\xi_{\mathbf{k}})}{v_F\delta q - iq_0}G(z, \mathbf{k})^2 + G(z, \mathbf{k})^2F(z)n_F(z)\right]
\end{aligned}
$$

where in the second equality we used the analyticity of F in the neighbourhood $z = \xi_{\mathbf{k}^\pm}$ and series expand in q. Likewise in the third equality we used the analyticity of the G's under the integral in the same neighbourhood of $z = \xi_{\mathbf{k}^\pm}$ to series expand to lowest order in q. For the penultimate equality we can consolidate the first two terms of the previous line into the whole contour integral $-\mathcal{C}$. Since this expansion holds within the linearized approximations, we can infer the OPE-like long distance expansion

$$G_n(ik_0^+,\mathbf{k}^+)G_n(ik_0^-,\mathbf{k}^-)F(ik_0) = \left(\frac{iq_0}{v_F\delta q - iq_0}\right)F(\xi_\mathbf{k})G_n(ik_0,\mathbf{k})^2 + G_n(ik_0,\mathbf{k})^2F(ik_0) \tag{A.34}$$

where the two sides are meant to agree only when, (i) inside the Matsubara frequency summation of ik_0, (ii) in the linearized approximation of the dispersion $\xi_\mathbf{k}^\pm$ and $n_F(\xi_{\mathbf{k}^\pm})$, (iii) to lowest most singular order in the small expansion of q and (iv) $F(z)$ is analytic in the neighbourhood of $z = \xi_\mathbf{k}$. Here we have also generalized the Green's function by a band index n to

$$G_n(z,\mathbf{k}) = \frac{1}{z - \xi_{n\mathbf{k}}} \tag{A.35}$$

and where the Fermi-velocity is $v_F\hat{\mathbf{k}} = \nabla_\mathbf{k}\xi_{n\mathbf{k}}|_{k_F}$. Note also both sides agree in the $q = 0$ limit only when the order of limits is taken such that $q_0 \to 0$ first then $|\mathbf{q}| \to 0$.

A.4.2 More gradient expansions

We can derive more intricate relations or q gradient expansions involving Green's functions which are valid only inside Matsubara frequency sums using the same methods as in the previous subsection. In a way, this is really a systematic way to perform the contour integral around each individual pole separately. The standard procedure is as follows:

0) Convert the Matsubara sum into a contour integral in the conventional way.

1) Deform and isolate the contour around the desired singularity eg. $z = \xi_{n\mathbf{k}^\pm}$.

2) Perform the contour and collect the residues just around that pole.

3) Using the assumed analyticity properties of the remaining factors of the integrand, Taylor expand about $q = 0$.

4) Re-organize terms such that the resulting terms may be re-incorporated into an integral using the original contour.

5) Identify terms inside to contour integral to derive the desired relation.

We list below several q gradient expansion "identities" that are derived using this technique:

$$[A] \quad G_n(ik_0^\pm, \mathbf{k}^\pm)^m F(ik_0) = G_n(ik_0, \mathbf{k})^m F(ik_0) \pm m \left(\frac{v_F \delta q - iq_0}{2} \right) G_n(ik_0, \mathbf{k})^{m+1} F(ik_0)$$

$$[B] \quad G_n(ik_0^+, \mathbf{k}^+) G_n(ik_0^-, \mathbf{k}^-) F(ik_0) = \left(\frac{iq_0}{v_F \delta q - iq_0} \right) F(\xi_\mathbf{k}) G_n(ik_0, \mathbf{k})^2 + G_n(ik_0, \mathbf{k})^2 F(ik_0)$$

$$
\begin{aligned}
[C] \quad & G_m(ik_0^+,\mathbf{k}^+)G_n(ik_0^-,\mathbf{k}^-)F(ik_0) \\
& = G_m(ik_0,\mathbf{k})G_n(ik_0,\mathbf{k})F(ik_0) \\
& + \left(\frac{iq_0}{\xi_{m\mathbf{k}}-\xi_{n\mathbf{k}}}\right)\left(\frac{G_m(ik_0,\mathbf{k})^2F(\xi_{m\mathbf{k}})+G_n(ik_0,\mathbf{k})^2F(\xi_{n\mathbf{k}})}{2}\right) \\
& + \left(\frac{v_F\delta q - iq_0}{\xi_{m\mathbf{k}}-\xi_{n\mathbf{k}}}\right)\frac{[G_m(ik_0,\mathbf{k})-G_n(ik_0,\mathbf{k})]^2}{2}F(ik_0)
\end{aligned}
$$

$$
\begin{aligned}
[D] \quad & G_m(ik_0^+,\mathbf{k}^+)G_m(ik_0^-,\mathbf{k}^-)F(ik_0) \\
& = G_m(ik_0,\mathbf{k})F(ik_0) - \frac{iq_0F(\xi_{m\mathbf{k}})}{(v_F\delta q - iq_0)^2}G_m(ik_0,\mathbf{k})^2 \\
& + \left(\frac{iq_0}{v_F\delta q - iq_0}\right)\left[\frac{F'(\xi_{m\mathbf{k}})G_m(ik_0,\mathbf{k})^2}{2} + F(\xi_{m\mathbf{k}})G_m(ik_0,\mathbf{k})^3\right]
\end{aligned}
$$

where in all cases $F(z)$ is assumed not to contain singularities where the G's are singular. Moreover we can always use these gradient expansion relations iteratively to derive more intricate expansions which we may encounter in higher order response bubble diagrams. Useful examples of these are listed below for quick reference

$$
[E] \quad G_n(k^+)G_n(k^-) = \left(\frac{v_F\delta q}{v_F\delta q - iq_0}\right)G_n(k)^2
$$

$$
\begin{aligned}
[F] \quad & G_m(k^+)G_n(k^-) \\
& = G_m(k)G_n(k) + \frac{v_F\delta q}{2}\frac{[G_m(k)-G_n(k)]^2}{(\xi_{m\mathbf{k}}-\xi_{n\mathbf{k}})} + \left(\frac{iq_0}{\xi_{m\mathbf{k}}-\xi_{n\mathbf{k}}}\right)G_m(k)G_n(k)
\end{aligned}
$$

$$[G]\quad G_m(k^{\pm})^2 G_n(k^{\mp})$$
$$= G_m(k)^2 G_n(k) \pm \frac{v_F\delta q - iq_0}{2}\left[\frac{G_m(k)[G_m(k)-G_n(k)][2G_m(k)-G_n(k)]}{\xi_{m\mathbf{k}}-\xi_{n\mathbf{k}}}\right]$$
$$\pm \frac{iq_0}{2(\xi_{m\mathbf{k}}-\xi_{n\mathbf{k}})}\left[2G_m(k)^3 - \left(\frac{G_n(k)^2+G_m(k)^2}{\xi_{m\mathbf{k}}-\xi_{n\mathbf{k}}}\right)\right]$$

$$[H]\quad G_n(k^+)G_n(k^-)G_m(k^+)$$
$$= \left(\frac{G_n(k)-G_m(k)}{\xi_{n\mathbf{k}}-\xi_{m\mathbf{k}}}\right)G_n(k) + \left(\frac{iq_0}{v_F\delta q - iq_0}\right)\frac{G_n(k)^2}{\xi_{n\mathbf{k}}-\xi_{m\mathbf{k}}}$$
$$- \frac{v_F\delta q - iq_0}{(\xi_{n\mathbf{k}}-\xi_{m\mathbf{k}})^3}[G_n(k)-G_m(k)] + \frac{v_F\delta q}{(\xi_{n\mathbf{k}}-\xi_{m\mathbf{k}})^2}\left(\frac{G_n(k)^2+G_m(k)^2}{2}\right)$$

$$[I]\quad G_m(k^+)G_m(k^-)G_n(k^-)$$
$$= \left(\frac{G_n(k)-G_m(k)}{\xi_{n\mathbf{k}}-\xi_{m\mathbf{k}}}\right)G_m(k) + \left(\frac{iq_0}{v_F\delta q - iq_0}\right)\frac{G_m(k)^2}{\xi_{m\mathbf{k}}-\xi_{n\mathbf{k}}}$$
$$+ \frac{v_F\delta q - iq_0}{(\xi_{n\mathbf{k}}-\xi_{m\mathbf{k}})^3}[G_n(k)-G_m(k)] - \frac{v_F\delta q}{(\xi_{n\mathbf{k}}-\xi_{m\mathbf{k}})^2}\left(\frac{G_n(k)^2+G_m(k)^2}{2}\right)$$

$$[J]\quad G_m(k^+)G_m(k^-)G_n(k^+)G_n(k^-)$$
$$= \left(\frac{iq_0}{v_F\delta q - iq_0}\right)\frac{G_m(k)^2+G_n(k)^2}{(\xi_{m\mathbf{k}}-\xi_{n\mathbf{k}})^2} + G_m(k)^2 G_n(k)^2$$

with the shorthand notation $G_n(k^\pm) \equiv G_n(ik_0^\pm, \mathbf{k}^\pm)$ and $G_n(k) \equiv G_n(ik_0, \mathbf{k})$. Also we can always factor products of G's with different band using the resolvent identity

$$G_n(k)G_m(k) \equiv \frac{G_n(k) - G_m(k)}{\xi_{n\mathbf{k}} - \xi_{m\mathbf{k}}}.$$

Finally, the expansions at small q lead to the factorization of the product of Green's functions into a factor that is $\mathbf{q}$ dependent and factor that is not. With an isotropic bare Fermi-surface, the $\mathbf{q}$-independent factor can be conveniently integrated over $\int \mathrm{d}|\mathbf{k}|$, with change of measure to an isotropic density of states $N(\xi)$. The remaining factor (the "Wilson coefficient" if you may) that is q dependent has to be integrated over $\mathbf{k}$-directions $\int \mathrm{d}\theta_{\mathbf{k}}$ and can often lead to complicated tensors with spatial indices. Techniques to perform this angular integration, use harmonic expansion presented in the appendix A.2.

A.5 Symmetries of Effective Actions

This is a short reminder of what to worry about when deriving effective actions. We start with the expectations for an effective field theory of a bosonic field ϕ_A which transforms as a representation of a group G. The index A is an representation index. The effect action for ϕ_A is generated from integrating high energy fluctuations of a more microscopic theory with group symmetry $\mathcal{G}$ which may be larger than G but must contain it $G \subset \mathcal{G}$. Schematically in the Euclidean space-time formulation

$$Z_0 \mathrm{e}^{-S_{\text{eff}}[\phi_A]} = \int \mathrm{d}[\Phi^<] \exp\left(-S_0[\phi_A] - S_1[\Phi^>] - S_2[\Phi^<, \phi_A]\right) \tag{A.36}$$

where $\Phi^>$ is collection of fast modes which may be a mix of fermionic and bosonic fields transforming under $\mathcal{G}$. They couple to ϕ_A as conjugate or dual representations of ϕ_A in G.

Note: Now usually the total high-energy action $S_0 + S_1 + S_2$ it taken to be invariant under $\mathcal{G}$ and hence G. This does not have to be case if there is an external perturbation or field that breaks the symmetry of G by picking a direction eg. a source term j^A or "mean-field" φ_A. Excluding this possibility, then the high-energy description must have the full $\mathcal{G}$ symmetry and $S_0 + S_1 + S_2$ must be $\mathcal{G}$ invariant and expressed entirely in $\Phi^<, \phi_A$ fields.

Next we limit to polynomial forms for the interaction terms, usually for power-counting and renormalizability reasons. Integrating out $\Phi^>$ then leads to the usual loop expansion and an effective action as a polynomial in ϕ_A and its derivatives

$$S_{\text{eff}}[\phi_A] = \int \left(S_1^A \phi_A + S_2^{AB} \phi_A \phi_B + S_3^{ABC} \phi_A \phi_B \phi_C + S_4^{ABCD} \phi_A \phi_B \phi_C \phi_D + \ldots\right) \tag{A.37}$$

here $S_n^{AB\cdots}$ are really kernels of linear operators on the $n-$field ϕ_A which may be non-local.[2] But we should really think of $S_n^{AB\cdots}$ as a tensor, in the algebraic sense, and because

[2] Can do an OPE to make them local in fancier treatments.

of bosonic symmetry as an element of $\mathrm{Symm}(\mathbb{V}^{*\otimes\ltimes})$. Here $\phi_A \in \mathbb{V}$ is a space-time field, transforming internally as a representation of G. That is $D : G \to \mathrm{End}(\mathbb{V}, \mathbb{V})$, specifies $D(g)$ as a matrix acting on $\mathbb{V}$. The $\mathrm{Symm}(\mathbb{V}^{*\otimes n})$ means the symmetrized tensors because ϕ_A is taken to be bosonic. If they were fermionic, which is possible, we would consider instead $S_n^{A_1,\ldots,A_2n} \in \bigwedge^{2n} \mathbb{V}^*$, the totally antisymmetric tensors of even degree.

We can then organize S_n for a given order n by Young Tableu since they are symmetric, for example in $n = 2$ The vertical tableu are excluded because they are anti-symmetric. For $S_2^{(AB)} = \frac{S_2^{AB}+S_2^{BA}}{2!} \rightsquigarrow \begin{array}{|c|c|}\hline A & B \\ \hline \end{array}$

general n we just have the single-row of n boxes.

Note that $S_n^{(A_1,\ldots,A_n)}$ still must transform as G representation. But since the microscopic theory is $G-$symmetric, the effective action S_{eff} must also respect this symmetry (before any spontaneous symmetry breaking). Which means that $S_n^{(A_1,\ldots,A_n)}$ must be proportional to or powers of invariant tensors of G, ie

$$S_n^{A_1,\cdots}\phi_{A_1}\ldots\phi_{A_n} = \sum_i \frac{\lambda_{ni}}{n!}\sigma_i^{A_1,\ldots,A_n}\phi_{A_1}\ldots\phi_{A_n} \tag{A.38}$$

where λ_{ni} is the "coupling" constant which in general can be a non-local operator, but it must be a G-invariant scalar. The symbols $\sigma_i^{A_1,\ldots,A_n}$ is the G-invariant symbol that it *totally symmetric* and is essentially determined from the representation theory of G. There may be more than one such invariant tensors. In this way symmetry entirely constrains the form of the effective action.

A.5.1 Symmetry of the quartic self-interaction Γ^4

The form of the lowest order expansion of $S_{\text{eff}}[\Gamma]$ can be determined from purely symmetric principles, in the case where gradient terms are first ignored. Here and throughout, Latin indices $i, j, k, l, \ldots = 1, 2$ will always refer to the quadrupolar modal indices $\sin 2\theta$ or $\cos 2\theta$

while Greek indices $\mu, \nu, \rho, \sigma, \ldots = x, y, z$ to the orbital mixing isospin-1 operator σ^x or σ^y. Note first that the global symmetry of the full microscopic fermionic action is $\mathbb{Z}_2 \times \mathrm{U}(1) \times \mathrm{O}(2) \cong \mathrm{O}(2) \times \mathrm{O}(2)$, and where the individual subgroups are described as follows.

1) The unitary U(1) stems from the *relative phase* of the ψ_1, ψ_2 fermions and is generated by the isospin rotation $\exp(i\phi\sigma^z/2)$, $\phi \in [0, 2\pi)$ which acts globally. Note that the orbital or full isospin symmetry of U(2) was broken down to U(1) by the choice of Hamiltonian by both the kinetic and interaction terms. In the kinetic term, this symmetry breaking is explicit due to the $\Delta\sigma^z$ term. There is also a separate U(1) gauge symmetry which stems from the common phase of $\psi_{1,2}$. However this just corresponds to the usual $\mathrm{U}(1)_{\mathrm{EM}}$ electromagnetic gauge symmetry which conserves total electric charge.

2) The O(2) group is the combined rotational and mirror symmetry in two dimensions. This is realized by the invariance of the Hamiltonian under the rotation of spatial axes $(x, y)^T \mapsto R \cdot (x, y)^T, R \in \mathrm{SO}(2)$ and the interchange of $\Phi_{\mu 1} \leftrightarrow \Phi_{\mu 2}$ or $\sin 2\theta_{\mathbf{k}} \leftrightarrow \cos 2\theta_{\mathbf{k}}$ which is implemented by a mirror operation in the xy-plane.

3) The $\mathbb{Z}_2$ factor corresponds to time-reversal which is implemented by just complex conjugation K and momentum inversion $\mathbf{k} \mapsto -\mathbf{k}$. In this case because the fermions are spinless, the time-reversal operator squares as $T^2 = +1$ with no Kramers' degeneracy. This $\mathbb{Z}_2$ may thought of as the "mirror" operation associated to the relative U(1) group, since $\mathbb{Z}_2 \times \mathrm{U}(1) \cong \mathrm{O}(2)$.

Having identified the group $\mathrm{O}(2) \times \mathrm{O}(2)$ as the microscopic symmetry, before any spontaneous symmetry breaking, it is clear what the transformation properties of the quadrupole density $\Phi_{\mu i}$ should be

$$\mathrm{U}(1): \quad \Phi_{\mu i} \mapsto S(\varphi)_{\mu\nu} \Phi_{\nu i} \;\; \text{where } S(\varphi) = \exp(i\varphi\sigma^z/2) \in \mathrm{SU}(2)$$

O(2): $\Phi_{\mu i}(\mathbf{k}) \mapsto R(2\vartheta)_{ij}\Phi_{\mu j}(R^T(\vartheta)\cdot\mathbf{k})$ where $R(2\vartheta), R(\vartheta)$ $\in$O(2) are induced by either a proper rotation

$\theta_{\mathbf{k}} \mapsto \theta_{\mathbf{k}} + \vartheta$, a reflection $\theta_{\mathbf{k}} \mapsto -\theta_{\mathbf{k}}$ or both.

$\mathbb{Z}_2$: $[\Phi_{xi}(\mathbf{r}), \Phi_{yi}(\mathbf{r})] \mapsto [\Phi_{xi}(\mathbf{r}), -\Phi_{yi}(\mathbf{r})]$ and in momentum space $[\Phi_{xi}(\mathbf{k}), \Phi_{yi}(\mathbf{k})] \mapsto [\Phi_{xi}(-\mathbf{k}), -\Phi_{yi}(-\mathbf{k}])$

The effective action is constructed from symmetric polynomials of $\Gamma_{\mu i}$ contracted with invariant tensors of O(2)×O(2). The invariant tensors of O(2) are δ, ϵ_{ij} and powers thereof. These second rank tensors correspond to taking dot ($\cdot$) and cross ($\times$) products on 2-vectors. Now δ_{ij} is reflection ($\mathbb{Z}_2$) invariant but $\epsilon_{ij} \to -\epsilon$ changes under reflection. Hence the only rank two O(2)×O(2) invariant tensor is $\delta_{ij}\delta_{\mu\nu}$. Written out this produces quadratic "mass" terms in the effective action,

$$I^{(2)}_{\nu i \mu j}\Gamma_{\mu i}\Gamma_{\nu j} = a\ \Gamma_{\mu i}\Gamma_{\mu i} = a(|\vec{\Gamma}_1|^2 + |\vec{\Gamma}_2|^2) \tag{A.39}$$

where

$$a = I^{(2)}_{x1x1} = I^{(2)}_{y1y1} = I^{(2)}_{x2x2} = I^{(2)}_{y2y2}. \tag{A.40}$$

For the 4th order monomial $I^{(4)}_{\mu i \nu j \sigma k \rho l}\Gamma_{\mu i}\Gamma_{\nu j}\Gamma_{\sigma k}\Gamma_{\rho l}$ we need the O(2)×O(2) invariant tensors of rank 4. Since there are no native (irreducible) invariant tensors of that rank, they have to be powers of the rank 2 ones. So we have

$$\delta_{\mu\nu}\delta_{ij}\delta_{\sigma\rho}\delta_{kl} \tag{A.41}$$

$$\epsilon_{\nu\nu}\epsilon_{ij}\epsilon_{\sigma\rho}\epsilon_{kl} \tag{A.42}$$

Thus we expect the expansion

$$I^{(4)}_{\mu i\nu j\sigma k\rho l} = c\delta_{\mu\nu}\delta_{ij}\delta_{\sigma\rho}\delta_{kl} + d\epsilon_{\nu\nu}\epsilon_{ij}\epsilon_{\sigma\rho}\epsilon_{kl} \tag{A.43}$$

where

$$c = I^{(4)}_{x1x1x1x1} = I^{(4)}_{x1x1y1y1} = \ldots = I^{(4)}_{y2y2y2y2} \tag{A.44}$$

$$d = I^{(4)}_{x1y2x1y2} = -I^{(4)}_{x1y2y1x2} = \ldots = I^{(4)}_{y1x2y1x2}. \tag{A.45}$$

Also these tensors contract to

$$\delta_{\mu\nu}\delta_{ij}\delta_{\sigma\rho}\delta_{kl}\ \Gamma_{\mu i}\Gamma_{\nu j}\Gamma_{\sigma k}\Gamma_{\rho l} = (\vec{\Gamma}_i \cdot \vec{\Gamma}_i)(\vec{\Gamma}_k \cdot \vec{\Gamma}_k) = (|\vec{\Gamma}_1|^2 + |\vec{\Gamma}_2|^2)^2 \tag{A.46}$$

$$\epsilon_{\mu\nu}\epsilon_{ij}\epsilon_{\sigma\rho}\epsilon_{kl}\ \Gamma_{\mu i}\Gamma_{\nu j}\Gamma_{\sigma k}\Gamma_{\rho l} = (\epsilon_{ij}\vec{\Gamma}_i \times \vec{\Gamma}_j)(\epsilon_{kl}\vec{\Gamma}_k \times \vec{\Gamma}_l) = 4(\vec{\Gamma}_1 \times \vec{\Gamma}_2)^2 \tag{A.47}$$

So

$$I^{(4)}_{\mu i\nu j\sigma k\rho l}\Gamma_{\mu i}\Gamma_{\nu j}\Gamma_{\sigma k}\Gamma_{\rho l} = c\ (|\vec{\Gamma}_1|^2 + |\vec{\Gamma}_2|^2)^2 + 4d\ (\vec{\Gamma}_1 \times \vec{\Gamma}_2)^2. \tag{A.48}$$

The coefficients a, c, d are determined from diagrammatic calculations.

A.6 More general gradient expansion of the susceptibility bubble

In this appendix we present a more general expression of the susceptibility bubble. Now the full diagram in Fig. 4.2 has the expression

$$I^{(2)}_{\mu i\nu j}(q) = \frac{-1}{k_F^4}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\,\frac{1}{\beta}\sum_{k_0}(\mathbf{k}^T\tau^i\mathbf{k})(\mathbf{k}^T\tau^j\mathbf{k})\left(\frac{1}{ik_0^+-\xi_{m\mathbf{k}^+}}\right)\left(\frac{1}{ik_0^- - \xi_{n\mathbf{k}^-}}\right) \tag{A.49}$$

$$= -\int \frac{\mathrm{d}^2k}{(2\pi)^2}\,(\hat{\mathbf{k}}^T\tau^i\hat{\mathbf{k}})(\hat{\mathbf{k}}^T\tau^j\hat{\mathbf{k}})\sum_{mn}\sigma^\mu_{mn}\sigma^\nu_{nm}\left[\frac{n_F(\xi_{n\mathbf{k}^-})-n_F(\xi_{m\mathbf{k}^+})}{iq_0+\xi_{n\mathbf{k}^-}-\xi_{m\mathbf{k}^+}}\right] \tag{A.50}$$

after summing over Matsubara frequencies of ik_0 and linearizing $\mathbf{k}\approx k_F\hat{\mathbf{k}}$ in the matrix elements. Then Taylor expanding $\xi_{m\mathbf{k}^\pm}$ and $n_F(\xi_{m\mathbf{k}^+ +})$ to linear order in $\mathbf{q}$,

$$\begin{aligned}
&I^{(2)}_{\mu i\nu j}(q)\\
&= -\int\frac{\mathrm{d}\theta_\mathbf{k}}{2\pi}\int \mathrm{d}\xi\ N(\xi)\ (\hat{\mathbf{k}}^T\tau^i\hat{\mathbf{k}})(\hat{\mathbf{k}}^T\tau^j\hat{\mathbf{k}})\\
&\quad\times\sum_{mn}\sigma^\mu_{mn}\sigma^\nu_{nm}\left[\frac{n_F(\xi_{n\mathbf{k}})-n_F(m\mathbf{k})-\frac{v_F\delta q}{2}\left(n'_F(\xi_{n\mathbf{k}})+n'_F(\xi_{m\mathbf{k}})\right)}{iq_0-v_F\delta q+\xi_{n\mathbf{k}}-\xi_{m\mathbf{k}}}\right]\\
&= -2r\int\frac{\mathrm{d}\theta_\mathbf{k}}{2\pi}\left[\frac{e_i(\theta_\mathbf{k})e_j(\theta_\mathbf{k})}{(iq_0-v_F\delta q)^2-(2v_F\Delta)^2}\right]\left\{\left[(2v_F\Delta)^2+v_F\delta q(iq_0-v_F\delta q)\left(\frac{\bar{r}}{r}\right)\right]\delta_{\mu\nu}\right.\\
&\qquad\left.+2v_F\Delta\left[(iq_0-v_F\delta q)+v_F\delta q\left(\frac{\bar{r}}{r}\right)\right]i\epsilon_{\mu\nu}\right\}
\end{aligned}$$

where the final line is yielded after enumerating the different $m,n = 1,2$ channels and some algebraic simplification. Also we have defined the $l=2$ angular harmonic functions $e_i(\theta_\mathbf{k}) := \hat{\mathbf{k}}^T\tau^i\hat{\mathbf{k}}$. Two interesting limits of this expression are

- $\Delta = 0$:

$$
\begin{aligned}
I^{(2)}_{\mu i \nu j}(q) &= -2\bar{r}\delta_{\mu\nu} \int \frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi}\, e_i(\theta_{\mathbf{k}}) e_j(\theta_{\mathbf{k}}) \left(\frac{v_F \delta q}{iq_0 - v_F \delta q} \right) \\
&= r\delta_{\mu\nu}\delta_{ij} - 2r \int \frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi}\, e_i(\theta_{\mathbf{k}}) e_j(\theta_{\mathbf{k}}) \left(\frac{iq_0}{iq_0 - v_F \delta q} \right) \delta_{\mu\nu}
\end{aligned}
$$

- $r = \bar{r}$:

$$
\begin{aligned}
I^{(2)}_{\mu i \nu j}(q) &= -2r \int \frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi} \frac{e_i(\theta_{\mathbf{k}}) e_j(\theta_{\mathbf{k}})(2v_F\Delta)}{(iq_0 - v_F\delta q)^2 - (2v_F\Delta)^2} \left\{ \left[(2v_F\Delta) + \left(\frac{v_F\delta q}{2v_F\Delta} \right)(iq_0 - v_F\delta q) \right] \delta_{\mu\nu} \right. \\
&\quad \left. + (iq_0) i\epsilon_{\mu\nu} \right\} \\
&= r\delta_{\mu\nu}\delta_{ij} - 2r(iq_0) \int \frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi} \frac{e_i(\theta_{\mathbf{k}}) e_j(\theta_{\mathbf{k}}) \left[(iq_0 - v_F\delta q)\delta_{\mu\nu} + (2v_F\Delta) i\epsilon_{\mu\nu} \right]}{(iq_0 - v_F\delta q)^2 - (2v_F\Delta)^2}
\end{aligned}
$$

where the second scenario $r = \bar{r}$ corresponds to a constant DOS or in 2D, a perfect quadratic dispersion relation.

An alternate expression that separates the static (q_0) and dynamic parts ($q_0 \neq 0$) is

$$
\begin{aligned}
I^{(2)}_{\mu i \nu j}(q) &= r\delta_{\mu\nu}\delta_{ij} \\
&\quad - 2r \int \frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi} \left[\frac{e_i(\theta_{\mathbf{k}}) e_j(\theta_{\mathbf{k}}) \left\{ (iq_0 - v_F\delta q)\delta_{\mu\nu} + (2v_F\Delta) i\epsilon_{\mu\nu} \right\} \left[iq_0 + \left(\frac{\bar{r}}{r} - 1 \right) v_F\delta q \right]}{(iq_0 - v_F\delta q)^2 - (2v_F\Delta)^2} \right].
\end{aligned}
$$

This expression is tremendously useful for reading off the gradient expansion terms. The relevant limit of interest would be $v_F\Delta \gg |q_0|, |v_F\delta q|$ such that we can safely expand the denominator in the integrand as

$$
\frac{1}{(2v_F\Delta)^2} \left(1 - \left(\frac{iq_0 - v_F\delta q}{2v_F\Delta} \right)^2 \right)^{-1} = \frac{1}{(2v_F\Delta)^2} \left(1 + \left(\frac{iq_0 - v_F\delta q}{2v_F\Delta} \right)^2 + \dots \right) \tag{A.51}
$$

$$
= \frac{1}{(2v_F\Delta)^2} + \left(\frac{v_F^2\delta q^2 - q_0^2 - 2iq_0 v_F\delta q}{(2v_F\Delta)^4} \right) + \dots \tag{A.52}
$$

So this produces gradient terms at 1st order and higher in q. Take going to second order and dropping terms which vanish under the angular integral, we have the gradient terms

$$
\begin{aligned}
I^{(2)}_{\mu i \nu j}(q) - I^{(2)}_{\mu i \nu j}(0) &= \frac{2r}{(2v_F\Delta)^2} \int \frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi}\, e_i(\theta_{\mathbf{k}}) e_j(\theta_{\mathbf{k}}) \left[(2v_F\Delta)(iq_0) i\epsilon_{\mu\nu} - q_0^2\delta_{\mu\nu} + \left(\frac{\bar{r}}{r} - 1\right) v_F^2 \delta q^2 \delta_{\mu\nu}\right] \\
&= r(i\epsilon_{\mu\nu}\delta)\left(\frac{iq_0}{2v_F\Delta}\right) + r\delta_{\mu\nu}\delta_{ij}\left(\frac{iq_0}{2v_F\Delta}\right)^2 + \delta_{ij}\delta_{\mu\nu}\left(\frac{\bar{r}-r}{4}\right)\left(\frac{|\mathbf{q}|^2}{2\Delta}\right)^2
\end{aligned}
$$

In real-space these lead to the following effective action gradient terms, which are local

$$
\delta\mathcal{L}^{\mathrm{grad}}_{\mathrm{eff}} = \frac{1}{2}\left(\frac{r}{2v_F\Delta}\right) i\epsilon_{\mu\nu}\epsilon_{ij}\Gamma_{\mu i}\partial_\tau\Gamma_{\nu i} + \frac{1}{2}\left[\frac{r}{(2v_F\Delta)^2}\right](\partial_\tau\Gamma_{\mu i})(\partial_\tau\Gamma_{\mu i}) + \frac{1}{2}\left[\frac{r-\bar{r}}{16\Delta^2}\right](\nabla\Gamma_{\mu i})\cdot(\nabla\Gamma_{\mu i}).
$$

The first term, which is the “Berry phase" term, is in important in giving the theory its $z = 1$ character. The $2v_F\Delta > 0$ gap is crucial in making the gradient terms local. In the other limit where $\Delta = 0$, the susceptibility kernel regains its non-local form

$$
\begin{aligned}
I^{(2)}_{\mu i \nu j}(q) - I^{(2)}_{\mu i \nu j}(0) &= 2r \int \frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi}\, e_i(\theta_{\mathbf{k}}) e_j(\theta_{\mathbf{k}}) \left(\frac{iq_0}{v_F\delta q - iq_0}\right)\delta_{\mu\nu} \\
&= -r\frac{|q_0|}{\|q\|}\left\{\delta_{ij} + \left(\frac{v_F|\mathbf{q}|}{\|q\| + |q_0|}\right)^4 \begin{pmatrix} \cos 4\theta_{\mathbf{q}} & \sin 4\theta_{\mathbf{q}} \\ \sin 4\theta_{\mathbf{q}} & -\cos 4\theta_{\mathbf{q}} \end{pmatrix}_{ij}\right\}\delta_{\mu\nu} \\
I^{(2)}_{\mu i \nu j}(0) &= r\delta_{ij}\delta_{\mu\nu}
\end{aligned}
$$

where $r = N(0)$ is the common DOS. This form of the susceptibility resembles the density-density response bubble and is non-local with the characteristic Landau damping form. Recall that

$$
\frac{|q_0|}{\|q\|} = \frac{|q_0|}{\sqrt{v_F^2|\mathbf{q}|^2 + q_0^2}} = \frac{|x|}{\sqrt{x^2+1}}, \qquad \text{where } x = \frac{q_0}{v_F|\mathbf{q}|}.
$$

A.7 Conventional Polarization Diagrams

Here we collect the polarization diagram calculations in (2+1) dimensions, which are standard bread and butter for Fermi-liquids. These are computed in the Euclidean signature momentum space or $(iq_0, \mathbf{q})$ in the common notation. However we include here the dispersions from the split Fermi-surfaces. The diagram in question is the first in the expansion of Figure. **??**. The Feynman rules (with the customary -1 for fermion loops) give for the density-density correlator

$$\Pi^{00}(q_0, \mathbf{q}) = (-e^2) \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \sum_{m=1}^{2} \left[\frac{n_F(\xi_{m\mathbf{k}^-}) - n_F(\xi_{m\mathbf{k}^+})}{iq_0 + (\xi_{m\mathbf{k}^-} - \xi_{m\mathbf{k}^+})} \right]. \tag{A.53}$$

One then proceeds customarily to linearize the dispersions as

$$\xi_{1\mathbf{k}} = v_F(|\mathbf{k}| - k_F + \Delta) \tag{A.54}$$

$$\xi_{2\mathbf{k}} = v_F(|\mathbf{k}| - k_F - \Delta) \tag{A.55}$$

to arrive at

$$\xi_{1\mathbf{k}^\pm} \approx v_F \left(|\mathbf{k}| \pm \tfrac{1}{2}\hat{\mathbf{k}} \cdot \mathbf{q} - k_F + \Delta \right) \tag{A.56}$$

$$\xi_{2\mathbf{k}^\pm} \approx v_F \left(|\mathbf{k}| \pm \tfrac{1}{2}\hat{\mathbf{k}} \cdot \mathbf{q} - k_F - \Delta \right) \tag{A.57}$$

$$\Rightarrow \xi_{m\mathbf{k}^-} - \xi_{m\mathbf{k}^+} \approx -v_F(\hat{k} \cdot \mathbf{q}) \tag{A.58}$$

Then expanding the Fermi functions yields by Taylor expansion

$$n_F(\xi_{1\mathbf{k}^\pm}) \approx \Theta(k_F - \Delta - |\mathbf{k}|) \mp \tfrac{1}{2}(\hat{\mathbf{k}} \cdot \mathbf{q})\ \delta(k_F - \Delta - |\mathbf{k}|) \tag{A.59}$$

$$n_F(\xi_{2\mathbf{k}^\pm}) \approx \Theta(k_F + \Delta - |\mathbf{k}|) \mp \tfrac{1}{2}(\hat{\mathbf{k}} \cdot \mathbf{q})\ \delta(k_F + \Delta - |\mathbf{k}|) \tag{A.60}$$

and taking their difference

$$n_F(\xi_{1\mathbf{k}^-}) - n_F(\xi_{1\mathbf{k}^+}) \approx (\hat{\mathbf{k}} \cdot \mathbf{q})\, \delta(k_F - \Delta - |\mathbf{k}|) \tag{A.61}$$

$$n_F(\xi_{2\mathbf{k}^-}) - n_F(\xi_{2\mathbf{k}^+}) \approx (\hat{\mathbf{k}} \cdot \mathbf{q})\, \delta(k_F + \Delta - |\mathbf{k}|) \tag{A.62}$$

which are expressions that commonly appear in these sorts of calculations. Note that these are purely Fermi-surface contributions, having support only at $k_F \pm \Delta$. In the more general calculation of $\langle \Gamma\Gamma \rangle$ susceptibilities, there are also contributions stemming from virtual excitation between bands which lie purely between Fermi-surfaces. Putting things together

$$\begin{aligned} \Pi^{00}(q_0, \mathbf{q}) &= (-e^2) \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \sum_{s=\pm} \left[\frac{(\hat{\mathbf{k}} \cdot \mathbf{q})\, \delta(k_F - s\Delta - |\mathbf{k}|)}{iq_0 - v_F(\hat{\mathbf{k}} \cdot \mathbf{q})} \right] \\ &= \frac{(-e^2)}{(2\pi)^2} \left(\frac{2k_F}{v_F} \right) \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \left[\frac{\cos\theta}{i\left(\frac{q_0}{v_F|\mathbf{q}|}\right) - \cos\theta} \right] \end{aligned} \tag{A.63}$$

after setting $\mathbf{q}$ parallel to the $+\hat{\mathbf{x}}$ without loss of generality (because of isotropy) and switching to polars. The integral can then be carried out conventionally, say by the substitution $z = \mathrm{e}^{i\theta}$ and performing a contour integral. Listed in Appendix **??** are some handy integrals that sometimes appear in Lindhard function calculations. Finally we get

$$\Pi^{00}(q_0, \mathbf{q}) = \frac{e^2}{(2\pi)^2} \left(\frac{2k_F}{v_F} \right) \left[1 - \frac{|x|}{\sqrt{x^2+1}} \right], \quad x := \frac{q_0}{v_F|q|}. \tag{A.64}$$

Note that $2k_F$ comes from the two Fermi-surfaces and not $2k_F$ scattering. The leading order constant can be understood as the 2D density-density static response of a free electron gas. Now the 2D density (per band) is $n_0 = \frac{k_F^2}{4\pi}$ which means

$$\Pi^{00}(0, |\mathbf{q}|) = \frac{2e^2}{\pi} \left(\frac{n_0}{E_F} \right), \quad E_F = v_F k_F. \tag{A.65}$$

Moving on to the transverse polarizations we have

$$\Pi^{ab}(q_0, \mathbf{q}) = \left(\frac{-e^2}{m^2}\right) \int \frac{\mathrm{d}^2 k}{(2\pi)^2} k_a k_b \sum_{m=1,2} \left[\frac{n_F(\xi_{m\mathbf{k}^-}) - n_F(\xi_{m\mathbf{k}^+})}{iq_0 + (\xi_{m\mathbf{k}^-} - \xi_{m\mathbf{k}^+})}\right] \tag{A.66}$$

and from the same linearization approximations of the dispersion we have

$$\Pi^{ab}(q_0, \mathbf{q}) = \left(\frac{-e^2}{m^2}\right) \int \frac{\mathrm{d}^2 k}{(2\pi)^2} |\mathbf{k}|^2 \sum_{s=\pm} \left[\frac{(\hat{\mathbf{k}} \cdot \mathbf{q})\ \delta(k_F - s\Delta - |\mathbf{k}|)}{iq_0 - v_F(\hat{\mathbf{k}} \cdot \mathbf{q})} d_a d_b\right] \tag{A.67}$$

where $d_1 := \cos\theta_{\mathbf{k}}$ and $d_2 := \sin\theta_{\mathbf{k}}$. Then going to polars and taking the $\int d|\mathbf{k}|$ integral gives

$$\begin{aligned}
\Pi^{ab}(q_0, \mathbf{q}) &= \left(\frac{-e^2}{4\pi^2 m^2}\right) \left[\sum_{s=\pm} (k_F - s\Delta)^3\right] \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi} \left[\frac{|\mathbf{q}| \cos\theta_{\mathbf{kq}}\ d_a(\theta_{\mathbf{k}}) d_b(\theta_{\mathbf{k}})}{iq_0 - v_F |\mathbf{q}| \cos\theta_{\mathbf{kq}}}\right] \\
&= \left(\frac{-e^2}{4\pi^2 m^2 v_F}\right) \left[2k_F^3 + 6\Delta^2 k_F\right] \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi} \left[\frac{\cos\theta_{\mathbf{k}}\ d_a(\theta_{\mathbf{k}} + \theta_{\mathbf{q}}) d_b(\theta_{\mathbf{k}} + \theta_{\mathbf{q}})}{ix - \cos\theta_{\mathbf{k}}}\right]
\end{aligned} \tag{A.68}$$

where $\cos\theta_{\mathbf{kq}} := \cos(\theta_{\mathbf{k}} - \theta_{\mathbf{q}}) = \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}$ and we have shifted variable of integration in the last line. The whole quantity is a 2nd-rank tensor and so we need only compute Π^{ab} with respect to $\hat{q}$ and its orthogonal direction. But under isotropic condition, expect that $\Pi^{ab} \propto \delta_{ab}$ and so we need only compute

$$\begin{aligned}
\Pi^{aa}(q_0, \mathbf{q}) &= \Pi^{11}(q_0, \mathbf{q}) + \Pi^{22}(q_0, \mathbf{q}) \\
&= \left(\frac{-2e^2 k_F}{4\pi^2 v_F}\right) \left(\frac{k_F^2 + 3\Delta^2}{m^2}\right) \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi} \frac{\cos\theta_{\mathbf{k}}}{ix - \cos\theta_{\mathbf{k}}} \\
&= \left(\frac{k_F^2 + 3\Delta^2}{m^2}\right) \Pi^{00}(q_0, \mathbf{q})
\end{aligned} \tag{A.69}$$

which gives

$$\Pi^{ab}(q_0,\mathbf{q}) = \left(\frac{k_F^2+3\Delta^2}{2m^2}\right)\Pi^{00}(q_0,\mathbf{q})\delta_{ab}$$
$$= \frac{1}{2}\left(v_F^2+\tfrac{3\Delta^2}{m^2}\right)\Pi^{00}(q_0,\mathbf{q})\delta_{ab} \qquad \text{(A.70)}$$

which is basically the density-density correlator $\times v_F^2\delta_{ab}$. The fact that it is symmetric tensor means that there is no Hall conductivity which is consistent with unbroken time-reversal symmetry.

It is common to express the Π tensors in terms of density of states $N(\xi)$ near the Fermi-energy. The density of states in general differs for the two bands. Re-doing the expression for the density-density correlator

$$\begin{aligned}\Pi^{00}(q_0,\mathbf{q}) &= (-e^2)\int\frac{\mathrm{d}^2k}{(2\pi)^2}\sum_{m=1,2}\left[\frac{\mathbf{q}\cdot(\nabla_{\mathbf{k}}\xi_{m\mathbf{k}})\,\delta(\xi_{m\mathbf{k}})}{iq_0-v_F(\hat{\mathbf{k}}\cdot\mathbf{q})}\right]\\ &= (-e^2)\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi}\left[N(v_F\Delta)+N(-v_F\Delta)\right]\left(\frac{|\mathbf{q}|v_F\cos\theta_{\mathbf{kq}}}{iq_0-v_F|\mathbf{q}|\cos\theta_{\mathbf{kq}}}\right)\\ &= (-e^2)\left[N(v_F\Delta)+N(-v_F\Delta)\right]\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_{\mathbf{k}}}{2\pi}\left(\frac{\cos\theta_{\mathbf{k}}}{ix-\cos\theta_{\mathbf{k}}}\right)\end{aligned} \qquad \text{(A.71)}$$

giving

$$\Pi^{00}(q_0,\mathbf{q}) = (-e^2)\left[N(v_F\Delta)+N(-v_F\Delta)\right]\left(1-\frac{|x|}{\sqrt{x^2+1}}\right) \qquad \text{(A.72)}$$
$$\Pi^{ab}(q_0,\mathbf{q}) = \frac{1}{2}v_F^2\Pi^{00}(q_0,\mathbf{q})\delta_{ab} \qquad \text{(A.73)}$$

in the linearized dispersion approximation, or alternative the large k_F and large v_F limit. The limit $\Delta\to 0$ yields the conventional polarizations of two degenerate Fermi-surfaces.

References

[Altland and Simons, 2010] Altland, A. and Simons, B. D. (2010). *Condensed matter field theory*. Cambridge University Press.

[Barci and Oxman, 2003] Barci, D. G. and Oxman, L. E. (2003). Strongly correlated fermions with nonlinear energy dispersion and spontaneous generation of anisotropic phases. *Phys. Rev. B*, 67:205108.

[Baym and Pethick, 1991] Baym, G. and Pethick, C. (1991). Landau fermi-liquid theory.

[Berry, 1985] Berry, M. (1985). Classical adiabatic angles and quantal adiabatic phase. *Journal of Physics A: Mathematical and General*, 18(1):15.

[Blount, 1962] Blount, E. (1962). Bloch electrons in a magnetic field. *Physical Review*, 126(5):1636.

[Castellani et al., 1996] Castellani, C., Di Castro, C., and Grilli, M. (1996). Non-fermi-liquid behavior and d-wave superconductivity near the charge-density-wave quantum critical point. *Zeitschrift für Physik B Condensed Matter*, 103(2):137–144.

[Castro Neto and Fradkin, 1994a] Castro Neto, A. H. and Fradkin, E. (1994a). Bosonization of fermi liquids. *Phys. Rev. B*, 49:10877–10892.

[Castro Neto and Fradkin, 1994b] Castro Neto, A. H. and Fradkin, E. (1994b). Bosonization of the low energy excitations of fermi liquids. *Phys. Rev. Lett.*, 72:1393–1397.

[Chakraborty, 1988] Chakraborty, T. (1988). P. pietilainen the fractional quantum hall effect.

[Fradkin, 2010] Fradkin, E. (2010). Electronic Liquid Crystal Phases in Strongly Correlated Systems. *ArXiv e-prints*.

[Fradkin, 2013] Fradkin, E. (2013). *Field theories of condensed matter physics*. Cambridge University Press.

[Fradkin et al., 2015] Fradkin, E., Kivelson, S. A., and Tranquada, J. M. (2015). Colloquium: Theory of intertwined orders in high temperature superconductors. *Reviews of Modern Physics*, 87(2):457.

[Haldane, 1981] Haldane, F. (1981). 'luttinger liquid theory'of one-dimensional quantum fluids. i. properties of the luttinger model and their extension to the general 1d interacting spinless fermi gas. *Journal of Physics C: Solid State Physics*, 14(19):2585.

[Haldane et al., 1987] Haldane, F., Prange, R., and Girvin, S. (1987). The quantum hall effect. *Springer Verlag, New York*.

[Haldane, 2004] Haldane, F. D. M. (2004). Berry curvature on the fermi surface: Anomalous hall effect as a topological fermi-liquid property. *Phys. Rev. Lett.*, 93:206602.

[Haldane, 2005] Haldane, F. D. M. (2005). Luttinger's Theorem and Bosonization of the Fermi Surface. *eprint arXiv:cond-mat/0505529*.

[Hertz, 1976] Hertz, J. A. (1976). Quantum critical phenomena. *Physical Review B*, 14(3):1165.

[Houghton et al., 2000] Houghton, A., Kwon, H.-J., and Marston, J. B. (2000). Multidimensional bosonization. *Advances in Physics*, 49(2):141–228.

[Houghton and Marston, 1993] Houghton, A. and Marston, J. B. (1993). Bosonization and fermion liquids in dimensions greater than one. *Phys. Rev. B*, 48:7790–7808.

[Jiang et al., 2009] Jiang, S., Xing, H., Xuan, G., Wang, C., Ren, Z., Feng, C., Dai, J., Xu, Z., and Cao, G. (2009). Superconductivity up to 30 k in the vicinity of the quantum critical point in bafe2 (as1- xpx) 2. *Journal of Physics: Condensed Matter*, 21(38):382203.

[Kwon et al., 1995] Kwon, H.-J., Houghton, A., and Marston, J. (1995). Theory of fermion liquids. *Physical Review B*, 52(11):8002.

[Lawler et al., 2006] Lawler, M. J., Barci, D. G., Fernández, V., Fradkin, E., and Oxman, L. (2006). Nonperturbative behavior of the quantum phase transition to a nematic fermi fluid. *Phys. Rev. B*, 73:085101.

[Löhneysen et al., 2007] Löhneysen, H. v., Rosch, A., Vojta, M., and Wölfle, P. (2007). Fermi-liquid instabilities at magnetic quantum phase transitions. *Rev. Mod. Phys.*, 79:1015–1075.

[Luther, 1976] Luther, A. (1976). Eigenvalue spectrum of interacting massive fermions in one dimension. *Phys. Rev. B*, 14:2153–2159.

[Maciejko et al., 2013] Maciejko, J., Hsu, B., Kivelson, S., Park, Y., and Sondhi, S. (2013). Field theory of the quantum hall nematic transition. *Physical Review B*, 88(12):125137.

[Mahan, 2013] Mahan, G. D. (2013). *Many-particle physics*. Springer Science & Business Media.

[Maslov and Chubukov, 2010] Maslov, D. L. and Chubukov, A. V. (2010). Fermi liquid near pomeranchuk quantum criticality. *Physical Review B*, 81(4):045110.

[Mattis, 1981] Mattis, D. (1981). The theory of magnetism, vol. i.

[Metzner et al., 1998] Metzner, W., Castellani, C., and Di Castro, C. (1998). Fermi systems with strong forward scattering. *Advances in Physics*, 47(3):317–445.

[Metzner et al., 2003] Metzner, W., Rohe, D., and Andergassen, S. (2003). Soft fermi surfaces and breakdown of fermi-liquid behavior. *Phys. Rev. Lett.*, 91:066402.

[Millis, 1993] Millis, A. (1993). Effect of a nonzero temperature on quantum critical points in itinerant fermion systems. *Physical Review B*, 48(10):7183.

[Nagaosa, 1999] Nagaosa, N. (1999). *Quantum field theory in strongly correlated electronic systems*. Springer Science & Business Media.

[Nagaosa et al., 2010] Nagaosa, N., Sinova, J., Onoda, S., MacDonald, A. H., and Ong, N. P. (2010). Anomalous hall effect. *Rev. Mod. Phys.*, 82:1539–1592.

[Oganesyan et al., 2001] Oganesyan, V., Kivelson, S. A., and Fradkin, E. (2001). Quantum theory of a nematic fermi fluid. *Phys. Rev. B*, 64:195109.

[Onoda and Nagaosa, 2003] Onoda, M. and Nagaosa, N. (2003). Quantized anomalous hall effect in two-dimensional ferromagnets: Quantum hall effect in metals. *Phys. Rev. Lett.*, 90:206601.

[Pomeranchuk, 1958] Pomeranchuk, I. Y. (1958). Zh. é ksp. teor. fiz. 35, 524 1958 sov. phys. *JETP*, 8:361.

[Pugh, 1930] Pugh, E. M. (1930). Hall effect and the magnetic properties of some ferromagnetic materials. *Phys. Rev.*, 36:1503–1511.

[Pugh and Lippert, 1932] Pugh, E. M. and Lippert, T. W. (1932). Hall e.m.f. and intensity of magnetization. *Phys. Rev.*, 42:709–713.

[Sachdev, 1999] Sachdev, S. (1999). Quantum phase transitions.

[Sebastian et al., 2010] Sebastian, S. E., Harrison, N., Altarawneh, M., Mielke, C., Liang, R., Bonn, D., and Lonzarich, G. (2010). Metal-insulator quantum critical point beneath the high tc superconducting dome. *Proceedings of the National Academy of Sciences*, 107(14):6175–6179.

[Slooten et al., 2009] Slooten, E., Naka, T., Gasparini, A., Huang, Y., and De Visser, A. (2009). Enhancement of superconductivity near the ferromagnetic quantum critical point in ucoge. *Physical review letters*, 103(9):097003.

[Stone, 1992] Stone, M. (1992). *Quantum Hall Effect*. World Scientific.

[Stoner, 1947] Stoner, E. C. (1947). Ferromagnetism. *Reports on Progress in Physics*, 11:43–112.

[Stormer, 1992] Stormer, H. (1992). Two-dimensional electron correlation in high magnetic fields. *Physica B: Condensed Matter*, 177(1-4):401–408.

[Sun and Fradkin, 2008] Sun, K. and Fradkin, E. (2008). Time-reversal symmetry breaking and spontaneous anomalous hall effect in fermi fluids. *Phys. Rev. B*, 78:245122.

[Thouless et al., 1982] Thouless, D. J., Kohmoto, M., Nightingale, M. P., and den Nijs, M. (1982). Quantized hall conductance in a two-dimensional periodic potential. *Phys. Rev. Lett.*, 49:405–408.

[Varma et al., 2002] Varma, C., Nussinov, Z., and Van Saarloos, W. (2002). Singular or non-fermi liquids. *Physics Reports*, 361(5):267–417.

[Wang et al., 2007] Wang, X., Vanderbilt, D., Yates, J. R., and Souza, I. (2007). Fermi-surface calculation of the anomalous hall conductivity. *Phys. Rev. B*, 76:195109.

[Wu et al., 2007] Wu, C., Sun, K., Fradkin, E., and Zhang, S.-C. (2007). Fermi liquid instabilities in the spin channel. *Phys. Rev. B*, 75:115103.

[Yosida, 1996] Yosida, K. (1996). *THEORY OF MAGNETISM.: Edition en anglais*, volume 122. Springer Science & Business Media.

[You and Fradkin, 2013] You, Y. and Fradkin, E. (2013). Field theory of nematicity in the spontaneous quantum anomalous hall effect. *Physical Review B*, 88(23):235124.